Section 10: Solutions

1. True or False?

(a) **True or False:** The probability of getting 20 heads in 100 independent tosses of a coin that has probability 5/6 of coming up heads is $(5/6)^{20}(1/6)^{80}$. **Solution:**

False. It is $\binom{100}{20}(5/6)^{20}(1/6)^{80}$.

(b) **True or False:** Suppose we roll a six-sided fair die twice independently. Then the event that the first roll is 3 and the sum of the two rolls is 6 are independent. **Solution:**

False. Let X_1 and X_2 be random variables that represent the values of the first and second rolls, respectively. $P(X_1 = 3) = \frac{1}{6}$. However, $P(X_1 = 3 \mid X_1 + X_2 = 6) = \frac{1}{5}$

(c) **True or False:** If X and Y are discrete, independent random variables, then so are X^2 and Y^2.

Solution:

True. X^2 and Y^2 are independent if $\mathbb{P}(X^2 = x, Y^2 = y) = \mathbb{P}(X^2 = x)\mathbb{P}(Y^2 = y)$.

$\mathbb{P}(X^2 = x, Y^2 = y) = \mathbb{P}(X = \sqrt{x}, Y = \sqrt{y})$ and since X and Y are independent:

$\mathbb{P}(X = \sqrt{x}, Y = \sqrt{y}) = \mathbb{P}(X = \sqrt{x})\mathbb{P}(Y = \sqrt{y}) = \mathbb{P}(X^2 = x)\mathbb{P}(Y^2 = y)$ Thus, X^2 and Y^2 are independent.

(d) **True or False:** The central limit theorem requires the random variables to be independent. **Solution:**

True. The central limit theorem requires the random variables to be i.i.d.

(e) **True or False:** Let A, B and C be any three events defined with respect to a probability space. Then $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A \cap B|C)\mathbb{P}(B|C)\mathbb{P}(C)$. **Solution:**

False. Suppose A, B, and C are all mutually independent, then

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) \neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(B)\,\mathbb{P}(C) = \mathbb{P}(A \cap B \mid C)\mathbb{P}(B \mid C)\mathbb{P}(C)$$

(f) **True or False:** Let A be the event that a random 5-card poker hand is a 4 of a kind (i.e. contains 4 cards of 1 rank and 1 card of a different rank) and let B be the event that it contains at least one pair. The events A and B are not independent. **Solution:**

True. A and B are independent if $\mathbb{P}(A, B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.
However,

$\mathbb{P}(A) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$

$\mathbb{P}(B) = 1 - \frac{\binom{13}{5}\times 4^5}{\binom{52}{5}}$, which is the probability of NOT getting all unique ranks in your hand (thus containing at least one pair)

$\mathbb{P}(A, B) = \mathbb{P}(A) = \frac{\binom{13}{1}\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \neq \mathbb{P}(A) \cdot \mathbb{P}(B)$

(g) **True or False:** If you flip a fair coin 1000 times, then the probability that there are 800 heads in total is the same as the probability that there are 80 heads in the first 100 flips. **Solution:**

false. Let X be the number of heads in 1000 flips of a fair coin, and Let Y be the number of heads in 100 flips of a fair coin.

$$\mathbb{P}(X = 800) = \binom{1000}{800} 0.5^{1000} = 6.17 \cdot 10^{-86} \neq 4.22 \cdot 10^{-10} = \binom{100}{80} 0.5^{100}$$

(h) **True or False:** If N is a nonnegative integer valued random variable, then

$$\mathbb{E}\left[\binom{N}{2}\right] = \binom{\mathbb{E}[N]}{2}.$$

Solution:

False. The left-hand side is

$$\mathbb{E}\left[\binom{N}{2}\right] = \mathbb{E}\left[\frac{N!}{(N-2)!\,2!}\right] = \frac{1}{2}\,\mathbb{E}\left[N^2 - N\right] = \frac{1}{2}\left(\mathbb{E}\left[N^2\right] - \mathbb{E}\left[N\right]\right)$$

while the right-hand side is

$$\binom{\mathbb{E}\left[N\right]}{2} = \frac{\mathbb{E}\left[N\right]!}{(\mathbb{E}\left[N\right] - 2)!\,2!} = \frac{1}{2}\,\left(\mathbb{E}\left[N\right]^2 - \mathbb{E}\left[N\right]\right)$$

and in general these equations are not equal because $\mathbb{E}\left[N^2\right] \neq \mathbb{E}\left[N\right]^2$

2. Short answer

(a) Consider a set S containing k distinct integers. What is the smallest k for which S is guaranteed to have 3 numbers that are the same mod 5? **Solution:**

$k = 11$. This is because modding any number by 5 yields 5 possible integers (i.e. slots). When distributing 11 numbers between these five slots, one slot must correspond to at least 3 integers mod 5.

(b) Let X be a random variable that can take any values between -10 and 10. What is the smallest possible value the variance of X can take? **Solution:**

0. This is because $Var(X) \geq 0$ and we can define the probability mass function in a way makes $Var(X) =$

0. For example, $p_X(x) = 1$ if $x = 7$ and 0 otherwise. Then we have

$$Var(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 = 7^2 - 7^2 = 0$$

(c) How many ways are there to rearrange the letters in the word KNICKKNACK? **Solution:**

$\frac{10!}{4!\,2!\,2!}$. Permute all 10 letters as if distinct, then divide by 4! to account for over counting the Ks; divide by 2! to account for over counting the Cs; and divide by 2! again to account for over counting the Ns

(d) I toss n balls into n bins uniformly at random. What is the expected number of bins with exactly k balls in them? **Solution:**

Let X be the number of bins with k balls in them. Let X_i be 1 if the ith bin has k balls in it, and otherwise 0. Note that $X = \sum_{i=1}^{n} X_i$. Since balls are distributed uniformly at random, the probability that a particular ball lands in a particular bin is $1/n$. Thus, the probability that k balls land in the ith bin is $\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k}$. By linearity of expectation we have

$$\mathbb{E}\left[X\right] = \sum_{i=1}^{n}\mathbb{E}\left[X_i\right] = \sum_{i=1}^{n}\mathbb{P}(X_i = 1) = \sum_{i=1}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} = n\binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k}$$

(e) Describe the probability mass function of a discrete distribution with mean 10 and variance 9 that takes only 2 distinct values. **Solution:**

Let X be a random variable that meets the above conditions. We can define the range and PMF as follows: $\Omega_X = \{a, b\}$, $\mathbb{P}(X = a) = 0.5$, and $\mathbb{P}(X = b) = 0.5$. This gives us two equations

$$\mathbb{E}\left[X\right] = 0.5a + 0.5b = 10 \rightarrow a = 20 - b$$

$$Var(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2 = 0.5a^2 + 0.5b^2 - (0.5a + 0.5b)^2 = 9$$

Combine the two equations to get

$$0.5(20 - b)^2 + 0.5b^2 - (0.5(20 - b) + 0.5b)^2 = 9$$

Solving for b gives us that $b = 13$ (I just plugged this directly into Wolfram Alpha). So, $a = 7$. Note that we could have chosen different probabilities, but choosing 0.5 for both makes solving the equations easier.

(f) Consider a six-sided die where $Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8$ and $Pr(5) = Pr(6) = 1/4$. Let X be the random variable which is the square root of the value showing. (For example, if the die shows a 1, X is 1, if the die shows a 2, X is $\sqrt{2}$, if the die shows a 3, $X = \sqrt{3}$ and so on.) What is the expected value of X? (Leave your answer in the form of a numerical sum; do not bother simplifying it.) **Solution:**

By the definition of expectation

$$\mathbb{E}\left[X\right] = \sum_{x=1}^{6}\sqrt{x}\,\mathbb{P}(x)$$

(g) A bus route has interarrival times that are exponentially distributed with parameter $\lambda = \frac{0.05}{\text{min}}$. What is the probability of waiting an hour or more for a bus? **Solution:**

Let X be an RV representing wait time, distributed according to $Exp(0.05)$

$$\mathbb{P}(X > 60) = 1 - F_X(60) = 1 - (1 - e^{-0.05 \cdot 60}) = 0.0498$$

(h) How many different ways are there to select 3 dozen colored roses if red, yellow, pink, white, purple and orange roses are available, and if roses of the same color are indistinguishable? **Solution:**

This is just a stars and bars problem. In this case there are 36 stars and $6 - 1 = 5$ bars. So there are $\binom{41}{5}$ ways to select 3 dozen roses.

(i) How many different ways are there to select 3 dozen colored roses if red, yellow, pink, white, purple and orange roses are available, such that there are at least 3 yellow and exactly 4 purple roses (roses of the same color are indistinguishable)? **Solution:**

This is still a stars and bars problem. We first take the 3 yellow and exactly 4 purple roses. Then, there are $36 - 3 - 4 = 29$ stars. There are $5 - 1 = 4$ bars because the remaining 29 roses can be any color except purple. So there are $\binom{29+4}{4}$ ways to select 3 dozen roses with the described constraints.

(j) Two identical 52-card decks are mixed together. How many distinct permutations of the 104 cards are there? **Solution:**

Perform the permutation as if it were 104 distinct items, and divide out the duplicates (each pair has $2!$ excess orderings, and there are 52 pairs), to get:

$$\frac{104!}{(2!)^{52}}$$

3. Random boolean formulas

Consider a boolean formula on n variables in 3-CNF, that is, conjunctive normal form with 3 literals per clause. This means that it is an "and" of "ors", where each "or" has 3 literals. Each parenthesized expression (i.e., each "or" of three literals) is called a clause. Here is an example of a boolean formula in 3-CNF, with $n = 6$ variables and $m = 4$ clauses.

$$(x_1 \vee x_3 \vee x_5) \wedge (\neg x_1 \vee \neg x_2 \vee x_6) \wedge (x_5 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_4 \vee x_5).$$

(a) What is the probability that $(\neg x_1 \vee \neg x_2 \vee x_3)$ evaluates to `true` if variable x_i is set to `true` with probability p_i, independently for all i? **Solution:**

$(\neg x_1 \vee \neg x_2 \vee x_3)$ is true when at least one of the following holds: $x_1 = \texttt{false}$, $x_2 = \texttt{false}$, $x_3 = \texttt{true}$. So we can write

$$\begin{aligned}\mathbb{P}((\neg x_1 \vee \neg x_2 \vee x_3) = \texttt{true}) &= \mathbb{P}(x_1 = \texttt{false} \cup x_2 = \texttt{false} \cup x_3 = \texttt{true}) \\ &= 1 - \mathbb{P}(x_1 = \texttt{true} \cap x_2 = \texttt{true} \cap x_3 = \texttt{false}) && \text{[Complementary probability]} \\ &= 1 - \mathbb{P}(x_1 = \texttt{true})\mathbb{P}(x_2 = \texttt{true})\mathbb{P}(x_3 = \texttt{false}) && \text{[Independence]} \\ &= 1 - x_1 \cdot x_2 \cdot (1 - x_3)\end{aligned}$$

(b) Consider a boolean formula in 3-CNF with n variables and m clauses. What is the expected number of satisfied clauses if each variable is set to `true` independently with probability 1/2? A clause is satisfied if it evaluates to `true`. (In the displayed example above, if $x_1, \ldots, x_5$ are set to `true` and x_6 is set to `false`, then all clauses but the second are satisfied.) **Solution:**

Let X be a random variable that represents the total number of satisfied clauses. Let X_i be a random variable that is 1 if the ith clause is satisfied, and otherwise 0. Note that $X = \sum_{i=1}^{m} X_i$. The $\mathbb{P}(X_i = 1) = 1 - 0.5^3$. This is because the ith clause is true when at least one of its disjuncts evaluates to true. As discussed in the previous part, this is equivalent to not all disjuncts evaluating to false. The probability that an individual disjuncts evaluates to false is 0.5, and because each conjuncts truth value is independent of the others, the probability that they are all false is 0.5^3. Using the complementary probability rule, we get $\mathbb{P}(X_i = 1) = 1 - 0.5^3$. By linearity of expectation

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \sum_{i=1}^{m} \mathbb{P}(X_i = 1) = \sum_{i=1}^{m} 1 - 0.5^3 = m\left(1 - 0.5^3\right)$$

4. Biased coin flips

We flip a biased coin with probability p of getting heads until we either get heads or we flip the coin three times. Thus, the possible outcomes of this random experiment are $< H >, < T, H >, < T, T, H >$ and $< T, T, T >$.

(a) What is the probability mass function of X, where X is the number of heads. (Notice that X is 1 for the first three outcomes, and 0 in the last outcome.) **Solution:**

Let E be an event that represents the outcome of our experiment. Note that E can take on four possible outcomes, however, they do not occur with equal probability.

$$\begin{aligned}\mathbb{P}(X = 0) &= \mathbb{P}(E =< T, T, T >) \\ &= (1 - p)^3 && \text{[Independent flips]}\end{aligned}$$

And

$$\begin{aligned}\mathbb{P}(X = 1) &= 1 - \mathbb{P}(X = 0) && \text{[Complementing]} \\ &= 1 - (1 - p)^3\end{aligned}$$

Alternatively, we can calculate $\mathbb{P}(X=1)$ as

$$\begin{aligned}\mathbb{P}(X=1) &= \mathbb{P}(E=<H> \cup E=<T,H> \cup E=<T,T,H>) \\ &= \mathbb{P}(E=<H>) + \mathbb{P}(E=<T,H>) + \mathbb{P}(E=<T,T,H>) && \text{[Disjoint events]} \\ &= p + (1-p)p + (1-p)^2 p && \text{[Independent flips]}\end{aligned}$$

Thus,

$$p_X(x) = \begin{cases} (1-p)^3, & x = 0 \\ p + (1-p)p + (1-p)^2 p, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

(b) What is the probability that the coin is flipped more than once? **Solution:**

The coin is flipped more than once if E is any of the last three outcomes. This is equivalent to E not being the first outcome. This occurs with probability $1 - \mathbb{P}(E=<H>) = 1 - p$.

(c) Are the events "there is a second flip and it is heads" and "there is a third flip and it is heads" independent? Justify your answer. **Solution:**

The event "there is a second flip and it is heads" is independent from the event "there is a third flip and it is heads" if and only if the following equation holds:

$$\mathbb{P}(E=<T,H>| \, E=<T,T,H>) = \mathbb{P}(E=<T,H>)$$

The LHS is 0 because it is impossible to flip T, H if you've already flipped T, T, H, whereas the RHS is $(1-p)p$. Therefore, the events are not independent.

(d) Given that we flipped more than once and ended up with heads, what is the probability that we got heads on the second flip? (No need to simplify your answer.) **Solution:**

Given that we flipped more than once and ended up with heads means that

$$E=<T,H> \cup E=<T,T,H>$$

Now, we are trying to find the following probability: $\mathbb{P}(E=<T,H>| \, (E=<T,H> \cup E=<T,T,H>))$. By the definition of conditional probability this is equal to

$$\begin{aligned}\frac{\mathbb{P}(E=<T,H> \cap (E=<T,H> \cup E=<T,T,H>))}{\mathbb{P}(E=<T,H> \cup E=<T,T,H>)} &= \frac{\mathbb{P}(E=<T,H>)}{\mathbb{P}(E=<T,H> \cup E=<T,T,H>)} \\ &= \frac{(1-p)p}{(1-p)p + (1-p)^2 p}\end{aligned}$$

The first equality holds because $E=<T,H>$ and $E=<T,T,H>$ are disjoint events, and the second equality holds from the probability values of the event E that we found in part (a).

5. Bitcoin users

There is a population of n people. The number of Bitcoin users among these n people is i with probability p_i, where, of course, $\sum_{0 \le i \le n} p_i = 1$. We take a random sample of k people from the population (without replacement). Use Bayes Theorem to derive an expression for the probability that there are i Bitcoin users in the population conditioned on the fact that there are j Bitcoin users in the sample. Let B_i be the event that there are i Bitcoin users in the population and let S_j be the event that there are j Bitcoin users in the sample. Your answer should be written in terms of the p_ℓ's, i, j, n and k.

Solution:

$$Pr(B_i|S_j) = \frac{Pr(S_j|B_i)Pr(B_i)}{Pr(S_j)} \qquad \text{by Bayes Theorem}$$

$$= \frac{\frac{\binom{i}{j}\binom{n-i}{k-j}}{\binom{n}{k}} \cdot p_i}{\sum_{\ell=0}^{n} Pr(S_j|B_\ell)Pr(B_\ell)} = \frac{\frac{\binom{i}{j}\binom{n-i}{k-j}}{\binom{n}{k}} \cdot p_i}{\sum_{\ell=0}^{n} \frac{\binom{\ell}{j}\binom{n-\ell}{k-j}}{\binom{n}{k}} \cdot p_\ell} = \frac{\binom{i}{j}\binom{n-i}{k-j} \cdot p_i}{\sum_{\ell=0}^{n} \binom{\ell}{j}\binom{n-\ell}{k-j} \cdot p_\ell}.$$

Above, we used the fact that $Pr(B_\ell) = p_\ell$ and the fact that $Pr(S_j|B_\ell)$ is the probability of choosing a subset of size k, where j of the selected people are from the subset of ℓ Bitcoin users and $k-j$ are from the remaining $n-\ell$ non-Bitcoin users.

6. Investments

You are considering three investments. Investment A yields a return which is X dollars where X is Poisson with parameter 2. Investment B yields a return of Y dollars where Y is Geometric with parameter 1/2. Investment C yields a return of Z dollars which is Binomial with parameters $n = 20$ and $p = 0.1$. The returns of the three investments are independent.

(a) Suppose you invest simultaneously in all three of these possible investments. What is the expected value and the variance of your total return?

(b) Suppose instead that you choose uniformly at random from among the 3 investments (i.e., you choose each one with probability 1/3). Use the law of total probability to write an expression for the probability that the return is 10 dollars. Your final expression should contain numbers only. No need to simplify your answer.

Solution:

(a) Let R be a random variable representing the total returns you get. If we invest in all of them simultaneously, then $R = X+Y+Z$. Then, $\mathbb{E}[R] = \mathbb{E}[X+Y+Z] = \mathbb{E}[X]+\mathbb{E}[Y]+\mathbb{E}[Z]$ by linearity of expectation.

Since X is Poisson with parameter 2, $\mathbb{E}[X] = 2$. Y is Geometric with parameter $\frac{1}{2}$, so $\mathbb{E}[Y] = \frac{1}{1/2} = 2$. Z is Binomial with parameters $n = 20$ and $p = 0.1$, so $\mathbb{E}[Z] = 20 \cdot 0.1 = 2$. Thus $\mathbb{E}[R] = 2+2+2 = 6$

$Var(R) = Var(X+Y+Z) = Var(X) + Var(Y) + Var(Z)$ because the returns from all three investments are independent. Because we know the distributions, we can read off their variances, with $Var(X) = \lambda = 2, Var(Y) = \frac{1-p}{p^2} = \frac{1/2}{1/4} = 2, Var(Z) = np(1-p) = 20 \cdot 0.1(0.9) = 1.8$.

Thus, $Var(R) = 2+2+1.8 = 5.8$

(b) Define events A, B, and C as randomly choosing Investments A, B, and C respectively. We want to find $\mathbb{P}(R = 10)$. We can break this up with the Law of Total Probability as

$$\mathbb{P}(R = 10) = \mathbb{P}(R = 10|A)(\frac{1}{3}) + \mathbb{P}(R = 10|B)(\frac{1}{3}) + \mathbb{P}(R = 10|C)(\frac{1}{3})$$

.

In each case, $R = X, Y$, or Z respectively, so we can plug in the PMFs of each function (and distribute out the $\frac{1}{3}$):

$$\mathbb{P}(R = 10) = \frac{1}{3}(e^{-2}\frac{2^{10}}{10!} + (0.5)^9 \cdot 0.5 + \binom{20}{10}0.1^{10}(0.9)^{10}) = 3.4040 \cdot 10^{-4}$$

7. Another continuous r.v.

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{when } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $E(X) = \frac{3}{5}$, find a and b.

Solution:

To find the value of two variables, we need two equations to solve as a system. We know that $\mathbb{E}[X] = \frac{3}{5}$, so we know, by the definition of expected value, that

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x) = \frac{3}{5}$$

Since $f(x)$ is defined to be 0 outside of the given range, we can integrate within only that range, plugging in $f(x)$:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x) = \int_{-\infty}^{0} xf(x) + \int_{0}^{1} xf(x) + \int_{1}^{\infty} xf(x) = \int_{0}^{1} x(a+bx^2) = \int_{0}^{1} ax+bx^3 = \frac{ax^2}{2} + \frac{bx^4}{4}\Big|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

We also know that a valid density function integrates to 1 over all possible values. Thus, we can perform the same process to get a second equation:

$$\int_{-\infty}^{\infty} f(x) = \int_{-\infty}^{0} xf(x) + \int_{0}^{1} xf(x) + \int_{1}^{\infty} xf(x) = \int_{0}^{1} (a + bx^2) = ax + \frac{bx^3}{3}\Big|_0^1 = a + \frac{b}{3} = 1$$

Solving this system of equations we get that $a = \frac{3}{5}, b = \frac{6}{5}$

8. Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together? **Solution:**

Apply the product rule. First order the families; there are are $n!$ ways to do this. Then consider the families one by one and reorder their members. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways to line these people up according to the given constraints.

9. Poisson CLT practice

Suppose $X_1, \ldots, X_n$ are iid $\mathsf{Poisson}(\lambda)$ random variables, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, the sample mean. How large should we choose n to be such that $\mathbb{P}(\frac{\lambda}{2} \leq \overline{X}_n \leq \frac{3\lambda}{2}) \geq 0.99$? Use the CLT and give an answer involving $\Phi^{-1}(\cdot)$. Then evaluate it exactly when $\lambda = 1/10$ using the Φ table on the last page.

Solution:

We know $\mathbb{E}[X_i] = \text{Var}(() X_i) = \lambda$. By the CLT, $\overline{X}_n \approx \mathcal{N}\left(\lambda, \frac{\lambda}{n}\right)$, so we can standardize this normal approximation.

$$\mathbb{P}(\frac{\lambda}{2} \leq \overline{X}_n \leq \frac{3\lambda}{2}) \approx \mathbb{P}(\frac{-\lambda/2}{\sqrt{\lambda/n}} \leq Z \leq \frac{\lambda/2}{\sqrt{\lambda/n}}) = \Phi\left(\frac{\lambda/2}{\sqrt{\lambda/n}}\right) - \Phi\left(\frac{-\lambda/2}{\sqrt{\lambda/n}}\right)$$

$$= \Phi\left(\frac{\lambda/2}{\sqrt{\lambda/n}}\right) - \left(1 - \Phi\left(\frac{\lambda/2}{\sqrt{\lambda/n}}\right)\right) = 2\Phi\left(\frac{\lambda/2}{\sqrt{\lambda/n}}\right) - 1 \geq 0.99 \to \Phi\left(\frac{\lambda/2}{\sqrt{\lambda/n}}\right) \geq 0.995$$

$$\to \frac{\sqrt{\lambda}}{2}\sqrt{n} \geq \Phi^{-1}(0.995) \to n \geq \frac{4}{\lambda}\left[\Phi^{-1}(0.995)\right]^2$$

We have $\lambda = \frac{1}{10}$ and from the table, $\Phi^{-1}(0.995) \approx 2.575$ so that $n \geq \frac{4}{1/10} \cdot 2.575^2 = 265.225$. So $n = 266$ is the smallest value that will satisfy the condition.

10. Law of Total Probability Review

(a) (Discrete version) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n+1$). Let H be the event that the coin comes up heads. What is $\mathbb{P}(H)$? **Solution:**

We can use the law of total probability, conditioning on $U = \frac{k}{n}$ for $k = 0, ..., n$. Note that the probability of getting heads conditioning on a fixed U value is U, and that the probability of U taking on any value in its range is $\frac{1}{n+1}$ since it is discretely uniform.

$$\mathbb{P}(H) = \sum_{k=0}^{n} \mathbb{P}(H|U = \frac{k}{n})\mathbb{P}(U = \frac{k}{n}) = \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)} \sum_{k=0}^{n} k = \frac{1}{n(n+1)} \frac{n(n+1)}{2} = \frac{1}{2}$$

(b) (Continuous version) Now suppose $U \sim$ Uniform(0,1) has the *continuous* uniform distribution over the interval $[0, 1]$. What is $\mathbb{P}(H)$? **Solution:**

We do the same thing, this time using the continuous law of total probability. Note, this time, that we're conditioning on $U = u$ and taking the integral with respect to u, and that the density of U for any value in its range is 1 because it is uniformly random.

$$\mathbb{P}(H) = \int_{-\infty}^{\infty} \mathbb{P}(H|U = u) f_U(u) du$$

We can take the integral from 0 to 1 instead because outside of that range the density of U is 0.

$$= \int_0^1 \mathbb{P}(H|U = u) f_U(u) du = \int_0^1 u \cdot 1 du = \frac{1}{2}[u^2]_0^1 = \frac{1}{2}$$

(c) Let's generalize the previous result we just used. Suppose E is an event, and X is a continuous random variable with density function $f_X(x)$. Write an expression for $\mathbb{P}(E)$, conditioning on X. **Solution:**

We use the continuous law of total probability again, this time not deriving it any further and sticking with negative infinity to infinity because we don't know the range of the RV X.

$$\mathbb{P}(E) = \int_{-\infty}^{\infty} \mathbb{P}(E|X = x) f_X(x) dx$$

11. A Red Poisson (From section 9 handout)

Suppose that $x_1, \ldots, x_n$ are i.i.d. samples from a Poisson(θ) random variable, where θ is unknown. Find the MLE of θ. **Solution:**

Because each Poisson RV is i.i.d., the likelihood of seeing that data is just the PMF of the Poisson distribution multiplied together for every x_i. From there, take the log-likelihood, then the first derivative, set it equal to 0 and solve for for $\hat{\theta}$.

$$\begin{aligned}
L(x_1, \ldots, x_n \mid \theta) &= \prod_{i=1}^{n} e^{-\theta} \frac{\theta^{x_i}}{x_i!} \\
\ln L(x_1, \ldots, x_n \mid \theta) &= \sum_{i=1}^{n} [-\theta - \ln(x_i!) + x_i \ln(\theta)] \\
\frac{\partial}{\partial \theta} \ln L(x_1, \ldots, x_n \mid \theta) &= \sum_{i=1}^{n} \left[-1 + \frac{x_i}{\theta}\right] \\
-n + \frac{\sum_{i=1}^{n} x_i}{\hat{\theta}} &= 0 \\
\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n}
\end{aligned}$$

12. Tail Bounds

Suppose $X \sim \mathsf{Bin}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds above, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

Solution:

We know that the expected value of a binomial distribution is np, so $\mathbb{P}(X \geq 4) \leq \mathbb{E}[X]/4 = 2.4/4 = 0.6$. We can use it since X is non-negative.

(b) Given an upper bound for this probability using Chebyshev's Inequality. You may have to rearrange algebraically and it may result in a weaker bound.

Solution:

We have

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6) \; .$$

Then, using Chebyshev's inequality,

$$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \mathsf{Var}(X)/1.6^2 = 1.44/1.6^2 = 0.5625 \; .$$

(c) Give an upper bound for this probability using the Chernoff Bound. **Solution:**

We seek to find $0 \leq \delta \leq 1$ such that $(1+\delta)\mathbb{E}[X] = 4$. Rearranging, we get that $\delta = 2/3$. Thus,

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1+2/3)2.4) \leq e^{-(2/3)^2\mathbb{E}[X]/3} = e^{-(4/9)\cdot 2.4\cdot(1/3)} \approx 0.7 \; .$$

(d) Give the exact probability.

Solution:

We have

$$\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) \; ,$$

which is

$$\binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}(0.4)^6(0.5)^0 \approx 0.1792 \; .$$

13. Covariance Connection

Let X be the network connection status, where $X = 0$ represents a stable connection and $X = 1$ represents an unstable connection. Let Y be the number of successes in data transmission, taking values in the set $\{0, 1, 2\}$. If $X = 0$, Y follows a Binomial distribution $\text{Bin}(2, 0.8)$, and if $X = 1$, Y follows a Binomial distribution $\text{Bin}(2, 0.3)$. The probabilities for X are given by $P(X = 0) = 0.8$ and $P(X = 1) = 0.2$. Find $\text{Cov}(X, Y)$. (note that we don't know that X and Y are independent here!)

Solution:

To calculate the covariance $\text{Cov}(X, Y)$, we need to determine $E[X]$, $E[Y]$, and $E[XY]$. The covariance is then given by the formula:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

First, we calculate $E[X]$: $E[X] = 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = 0 \cdot 0.8 + 1 \cdot 0.2 = 0.2$

Next, we calculate $E[Y]$. First, we calculate $E[Y \mid X = 0]$ and $E[Y \mid X = 1]$. Based on what's given in the problem and using the formula for expectation for a binomial: $E[Y \mid X = 0] = 2 \cdot 0.8 = 1.6$ and $E[Y \mid X = 1] = 2 \cdot 0.3 = 0.6$. Using the law of total expectation:

$$E[Y] = E[Y \mid X = 0]P(X = 0) + E[Y \mid X = 1]P(X = 1) = 1.6 \cdot 0.8 + 0.6 \cdot 0.2 = 1.4$$

To compute $E[XY]$, we first construct the joint PMF for XY and then use the definition of expectation. The possible values for XY are 0, 1, and 2. Let's compute the probabilities for each value:

$$\begin{aligned} P(XY = 0) &= P(X = 0 \cup Y = 0) = P(X = 0) + P(Y = 0) - P(X = 0 \cap Y = 0) \\ &= P(X = 0) + P(Y = 0) - P(X = 0)P(Y = 0 \mid X = 0) = 0.8 + 0.13 - 0.8 \cdot 0.2^2 = 0.898 \end{aligned}$$

$$P(XY = 1) = P(X = 1 \cap Y = 1) = P(X = 1)P(Y = 1 \mid X = 1) = 0.2 \cdot (2 \cdot 0.3 \cdot 0.7) = 0.084$$

$$P(XY = 2) = P(X = 1 \cap Y = 2) = P(X = 1)P(Y = 2 \mid X = 1) = 0.2 \cdot (0.3^2) = 0.018$$

In the above calculations, we use that $P(Y = 0) = P(Y = 0|X = 0)P(X = 0) + P(Y = 0|X = 1)P(X = 1) = 0.2^2 \cdot 0.8 + 0.7^2 \cdot 0.2 = 0.13$ Now, using the definition of expectation, we have:

$$E[XY] = 0 \cdot 0.898 + 1 \cdot 0.084 + 2 \cdot 0.018 = 0.12$$

Therefore, $E[XY] = 0.12$. Finally, we calculate the covariance Cov(X, Y):

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 0.12 - 0.2 \cdot 1.4 = -0.16$$

Therefore, the covariance Cov(X, Y) is -0.16. The negative covariance of -0.16 between the network connection status X and the number of successes in data transmission Y indicates an inverse relationship, suggesting that as the network connection status becomes less stable (i.e., as $X = 1$), the likelihood of successess in data transmission decreases, and vice versa, as expected!