# Section 8: Solutions

## Review of Main Concepts

- **Multivariate: Discrete to Continuous:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$ | $f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$ |
| **Joint range/support** $\Omega_{X,Y}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$ |
| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$ |
| **Expectation** | $\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$ | $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$ |
| **Independence** must have | $\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ | $\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ |

- **Law of Total Probability (r.v. version):** If $X$ is a discrete random variable, then

$$\mathbb{P}(A) = \sum_{x \in \Omega_X} \mathbb{P}(A | X = x) p_X(x) \qquad \text{discrete } X$$

- **Law of Total Expectation (Event Version)**: Let $X$ be a discrete random variable, and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X | A_i] \mathbb{P}(A_i)$$

- **Conditional Expectation**: See table. Note that linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y | A] = \mathbb{E}[X | A] + \mathbb{E}[Y | A]$
- **Law of Total Expectation (RV Version)**: Suppose $X$ and $Y$ are random variables. Then,

$$\mathbb{E}[X] = \sum_{y} \mathbb{E}[X | Y = y] p_Y(y) \qquad \text{discrete version.}$$

- **Conditional distributions**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Conditional PMF/PDF** | $p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ | $f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ |
| **Conditional Expectation** | $\mathbb{E}[X\|Y=y] = \sum_x x p_{X\|Y}(x\|y)$ | $\mathbb{E}[X\|Y=y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$ |

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A | X = x) f_X(x) dx$$

- **Continuous Law of Total Expectation:**

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X | Y = y] f_Y(y) dy$$

- **Markov's Inequality**: Let $X$ be a non-negative random variable, and $\alpha > 0$. Then,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

- **Chebyshev's Inequality**: Suppose $Y$ is a random variable with $\mathbb{E}[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$,

$$\mathbb{P}(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$

- **(Multiplicative) Chernoff Bound**: Let $X_1, X_2, ..., X_n$ be *independent* Bernoulli random variables.

  Let $X = \Sigma_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $0 \leq \delta \leq 1$,

  - $\mathbb{P}(X \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$
  - $\mathbb{P}(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$

## 1. Content Review

(a) True or false: the Union Bound always gives a result in $[0, 1]$.
**Solution:**

False. Consider $X$ and $Y$, which are independent indicator random variables.

Suppose $p_X(x) = \begin{cases} 0.75 & x = 0 \\ 0.25 & x = 1 \end{cases}$ and $p_Y(y) = \begin{cases} 0.75 & y = 0 \\ 0.25 & y = 1 \end{cases}$.

Then we may apply the Union Bound to place a bound on $P(X = 0 \cup Y = 0)$:

$$P(X = 0 \cup Y = 0) \leq P(X = 0) + P(Y = 0) = 0.75 + 0.75 = 1.5.$$

In these cases, the Union Bound tells us very little, since the probability of any event occurring is at most 1.

(b) True or false: Markov's Inequality always gives a non-negative result.
**Solution:**

True. Markov's Inequality is

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

as long as $X$ is a non-negative random variable and $\alpha > 0$. Since $X$ is a non-negative random variable, $\mathbb{E}[X] \geq 0$, so $\frac{\mathbb{E}[X]}{\alpha} \geq 0$.

(c) Suppose $C$ and $D$ are discrete random variables. Then $\mathbb{E}[C|D = d] =$

- ◯ $\sum_d d p_{D|C}(d|c)$
- ◯ $\sum_c c p_{C|D}(c|d)$
- ◯ $\int_{-\infty}^{\infty} c f_{c|d} dx$
- ◯ $\frac{\mathbb{E}[C]}{\mathbb{E}[D]}$

**Solution:**

Choice b is the correct answer from the definition of conditional expectation for discrete random variables.

(d) Suppose $X$ and $Y$ are random variables and $A$ is an event. Given that $\mathbb{E}[X|A] = 4$ and $\mathbb{E}[Y|A] = 10$, what is $\mathbb{E}[2X + Y/2|A]$?

- ◯ 14
- ◯ 18

◯ 9

◯ 13

**Solution:**

Choice d is the correct answer since linearity of expectation still applies to conditional expectation:

$$\mathbb{E}[2X + Y/2|A] = \mathbb{E}[2X|A] + \mathbb{E}[Y/2|A] = 2\mathbb{E}[X|A] + \mathbb{E}[Y|A]/2 = 2 \cdot 4 + 10/2 = 8 + 5 = 13.$$

(e) True or false: Chebyshev's Inequality can best be described as giving an upper bound on the distribution's right tail.
**Solution:**

False. Chebyshev's Inequality gives an upper bound on the sum of the probabilties of the left and right tails of the distribution.

## 2. Tail bounds

Suppose $X \sim \mathsf{Binomial}(6, 0.4)$. We will bound $\mathbb{P}(X \geq 4)$ using the tail bounds we've learned, and compare this to the true result.

(a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?
**Solution:**

We know that the expected value of a binomial distribution is $np$, so: $\mathbb{P}(X \geq 4) \leq \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since $X$ is nonnegative.

(b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound. **Solution:**

$\mathbb{P}(X \geq 4) = \mathbb{P}(X - 2.4 \geq 1.6) \leq \mathbb{P}(|X - 2.4| \geq 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geq 1.6$. Then, using Chebyshev's inequality we get:
$\mathbb{P}(|X - 2.4| \geq 1.6) \leq \frac{Var(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$

(c) Give an upper bound for this probability using the Chernoff bound. **Solution:**

First, we solve for the values of $\delta$ that will allow us to use the Chernoff bound. We want $(1 + \delta)E[X] = (1 + \delta)2.4 = 4$. Solving for $\delta$ here gives use $\delta = \frac{2}{3}$. Now, we can directly plug into the Chernoff bound. $\mathbb{P}(X \geq 4) = \mathbb{P}(X \geq (1 + \frac{2}{3})2.4) \leq e^{-(\frac{2}{3})^2\mathbb{E}[X]/3} = e^{-4\times2.4/27} \approx 0.7$

(d) Give the exact probability. **Solution:**

Since $X$ is a binomial, we know it has a range from 0 to $n$ (or in this case 0 to 6). Thus, the possible values to satisfy $X \geq 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geq 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

## 3. Exponential Tail Bounds

Let $X \sim \text{Exp}(\lambda)$ and $k > 1/\lambda$.

(a) Use Markov's inequality to bound $\text{P}(X \geq k)$.

**Solution:**

We can use Markov's inequality here because $X$ is non-negative since it is an exponential distribution. We also know that $E[X] = 1/\lambda$ because $X \sim \text{Exp}(\lambda)$. By Markov's inequality, we get that:

$$\mathbb{P}(X \geq k) \leq \frac{1}{\lambda k}$$

(b) Use Markov's inequality to bound $\text{P}(X < k)$. **Solution:**

From Markov's inequality (and our answer in (a)), we know that $P(X \geq k) \leq \frac{1}{\lambda k}$. Then,

$$\begin{aligned}
P(X \geq k) &\leq \frac{1}{\lambda k} \\
-P(X \geq k) &\geq -\frac{1}{\lambda k} && \text{multiplying be a negative flips the inequality} \\
1 - P(X \geq k) &\geq 1 - \frac{1}{\lambda k} \\
P(X < k) &\geq 1 - \frac{1}{\lambda k} && \text{by definition of complement}
\end{aligned}$$

Note that because we took the complement and the sign flipped, we have now found a *lower* bound for $P(X < k)$.

(c) Use Chebyshev's inequality to bound $\text{P}(X \geq k)$. **Solution:**

We rearrange algebraically to get into the form to apply Chebyshev's inequality. We then plug in the corresponding values and $Var(X) = \frac{1}{\lambda^2}$.

$$\mathbb{P}(X \geq k) = \mathbb{P}\left(X - \frac{1}{\lambda} \geq k - \frac{1}{\lambda}\right) \leq \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \geq k - \frac{1}{\lambda}\right) \leq \frac{1}{\lambda^2(k - 1/\lambda)^2} = \frac{1}{(\lambda k - 1)^2}$$

(d) What is the exact formula for $\text{P}(X \geq k)$? **Solution:**

Using the CDF for an exponential distribution and definition of complement:

$$\mathbb{P}(X \geq k) = 1 - P(X \leq k) = 1 - (1 - e^{-\lambda k}) = e^{-\lambda k}$$

(e) For $\lambda k \geq 3$, how do the bounds given in parts (a), (c), and (d) compare?

**Solution:**

$$e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$$

so Markov's inequality gives the worst bound.

## 4. Robbie's Late!

Suppose the probability Robbie is late to teaching lecture on a given day is at most 0.01. Do not make any independence assumptions.

(a) Use a Union Bound to bound the probability that Robbie is late at least once over a 30-lecture quarter. **Solution:**

Let $R_i$ be the event Robbie is late to lecture on day $i$ for $i = 1, ..., 30$. Then, by the union bound,

$$\begin{aligned}
\mathbb{P}(\text{late at least once}) &= \mathbb{P}(\bigcup_{i=1}^{30} R_i) \\
&\leq \sum_{i=1}^{30} \mathbb{P}(R_i) && [\text{union bound}] \\
&\leq \sum_{i=1}^{30} 0.01 && [\mathbb{P}(R_i) \leq 0.01] \\
&= 0.30
\end{aligned}$$

(b) Use a Union Bound to bound the probability that Robbie is **never** late over a 30-lecture quarter. **Solution:**

As in the previous part, let $R_i$ be the event Robbie is late to lecture on day $i$ for $i = 1, ..., 30$. Then, by the union bound, we found that

$$\mathbb{P}(\text{late at least once}) \leq 0.30$$

The probability Robbie is never late is the complement of the probability he is late at least once over the

30 lectures. Taking the complement and doing algebra:

$$\begin{aligned} \mathbb{P}(\text{late at least once}) &\leq 0.30 \\ -\mathbb{P}(\text{late at least once}) &\geq -0.30 && \text{[multiplying by negative flips the inequality]} \\ 1 - \mathbb{P}(\text{late at least once}) &\geq 1 - 0.30 \\ \mathbb{P}(\text{never late}) &\geq 0.70 \end{aligned}$$

Note that we have now found a *lower* bound for this probability using the union bound because of taking the complement.

(c) Use a Union Bound to bound the probability that Robbie is late at least once over a 120-lecture quarter. **Solution:**

Let $R_i$ be the event Robbie is late to lecture on day $i$ for $i = 1, ..., 120$. Then, by the union bound,

$$\begin{aligned} \mathbb{P}(\text{late at least once}) &= \mathbb{P}(\bigcup_{i=1}^{120} R_i) \\ &\leq \sum_{i=1}^{120} \mathbb{P}(R_i) && \text{[union bound]} \\ &\leq \sum_{i=1}^{120} 0.01 && [\mathbb{P}(R_i) \leq 0.01] \\ &= 1.20 \end{aligned}$$

Notice that $\mathbb{P}(\text{late at least once}) \leq 1.20$ is not a very helpful bound since probabilities have to be at most 1 already.

## 5. Trinomial Distribution

A generalization of the Binomial model is when there is a sequence of $n$ independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1, 2, 3$ and of course $p_1 + p_2 + p_3 = 1$. Let $X_i$ be the number of times outcome $i$ occurred for $i = 1, 2, 3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$. **Solution:**

We use a similar argument as for the binomial PMF. $\binom{n}{x_1,x_2,x_3}$ is the number of ways to select which of the $n$ outcomes result in each of the 3 outcomes. Then, we multiply the probabilities of each trial being the corresponding outcome (e.g., $p_1^{x_1}$ is the probability that all $x_1$ trials end up being outcome 1). This gives use the following PMF:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \binom{n}{x_1, x_2, x_3} \prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

## 6. Do You "Urn" to Learn More About Probability?

Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i = 1$ if the $i$-th ball selected is white and let it be equal to 0 otherwise. Give the joint probability mass function of

(a) $X_1, X_2$ **Solution:**

Here is one way of defining the joint pmf of $X_1, X_2$

$$\mathbb{P}(X_1 = 1, X_2 = 1) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 1 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{20}{156}$$

$$\mathbb{P}(X_1 = 1, X_2 = 0) = \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 0 \mid X_1 = 1) = \frac{5}{13} \cdot \frac{8}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 1) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 1 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{5}{12} = \frac{40}{156}$$

$$\mathbb{P}(X_1 = 0, X_2 = 0) = \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0 \mid X_1 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{56}{156}$$

(b) $X_1, X_2, X_3$ **Solution:**

Instead of listing out all the individual probabilities, we could write a more compact formula for the pmf. In this problem, the denominator is always $P(13, k)$, where $k$ is the number of random variables in the joint pmf. And the numerator is $P(5, i)$ times $P(8, j)$ where $i$ and $j$ are the number of 1s and 0s, respectively.

If we wish to compute $p_{X_1,X_2,X_3}(x_1, x_2, x_3)$, then the number of 1s (i.e., white balls) is $x_1 + x_2 + x_3$, and the number of 0s (i.e., red balls) is $(1 - x_1) + (1 - x_2) + (1 - x_3)$. Then, we can write the pmf as follows:

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3) = \frac{10!}{13!} \cdot \frac{5!}{(5 - x_1 - x_2 - x_3)!} \cdot \frac{8!}{(5 + x_1 + x_2 + x_3)!}$$

## 7. Successes

Consider a sequence of independent Bernoulli trials, each of which is a success with probability $p$. Let $X_1$ be the number of failures preceding the first success, and let $X_2$ be the number of failures between the first 2 successes. Find the joint pmf of $X_1$ and $X_2$. Write an expression for $E[\sqrt{X_1 X_2}]$. You can leave your answer in the form of a sum. **Solution:**

$X_1$ and $X_2$ take on two particular values $x_1$ and $x_2$, when there are $x_1$ failures followed by one success, and then $x_2$ failures followed by one success. Since the Bernoulli trials are independent the joint pmf is

$$p_{X_1,X_2}(x_1, x_2) = (1 - p)^{x_1} p \cdot (1 - p)^{x_2} p = (1 - p)^{x_1 + x_2} p^2$$

for $(x_1, x_2) \in \Omega_{X_1,X_2} = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$. By the definition of expectation

$$E[\sqrt{X_1 X_2}] = \sum_{(x_1,x_2)\in\Omega_{X_1,X_2}} \sqrt{x_1 x_2} \cdot (1 - p)^{x_1 + x_2} p^2.$$

## 8. Continuous joint density

The joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of $W$ and $V$ is given by

$$f_{W,V}(w, v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent? Are $W$ and $V$ independent?

**Solution:**

For two random variables $X, Y$ to be independent, we must have $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X,\ y \in \Omega_Y$. Let's start with $X$ and $Y$ by finding their marginal PDFs. By definition, and using the fact that the joint PDF is 0 outside of $y > 0$, we get:

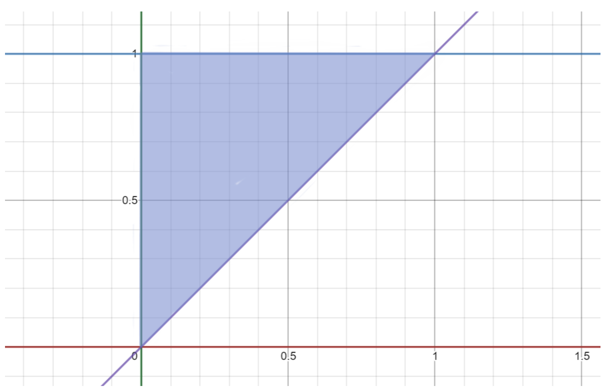
$$f_X(x) = \int_0^\infty xe^{-(x+y)}dy = e^{-x}x$$

We do the same to get the PDF of $Y$, again over the range $x > 0$:

$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, $X$ and $Y$ are independent.

We can see that $W$ and $V$ are not independent simply by observing that $\Omega_W = (0,1)$ and $\Omega_V = (0,1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product. Specifically, looking at their range of $f_{W,V}(w,v)$. Graphing it with w as the "x-axis" and v as the "y-axis", we see that :




The shaded area is where the joint pdf is strictly positive. Looking at it, we can see that it is not rectangular, and therefore it is not the case that $\Omega_{W,V} = \Omega_W \times \Omega_V$. Remember, the joint range being the Cartesian product of the marginal ranges is not sufficient for independence, but it is *necessary*. Therefore, this is enough to show that they are not independent.

## 9. Trapped Miner

A miner is trapped in a mine containing 3 doors.

- $D_1$: The $1^{st}$ door leads to a tunnel that will take him to safety after 3 hours.
- $D_2$: The $2^{nd}$ door leads to a tunnel that returns him to the mine after 5 hours.
- $D_3$: The $3^{rd}$ door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

**Solution:**

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let $D_i$ be the event the $i^{th}$ door is chosen. $i \in \{1, 2, 3\}$. Finally, let $T_3$ be the time it takes to return to the mine in the third case only (a random variable). Note that the expectation of $T_3$ is $12 * \frac{1}{3}$ because it is binomially distributed with parameters $n = 12, p = \frac{1}{3}$. By Law of Total Expectation, linearity of expectation, and by applying the conditional expectations given by the problem statement:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T|D_1]\,\mathbb{P}(D_1) + \mathbb{E}[T|D_2]\,\mathbb{P}(D_2) + \mathbb{E}[T|D_3]\,\mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3 + T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (\mathbb{E}[T_3] + \mathbb{E}[T]) \cdot \frac{1}{3} \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3}
\end{aligned}$$

Solving this equation for $\mathbb{E}[T]$, we get

$$\mathbb{E}[T] = 12$$

Therefore, the expected number of hours for this miner to reach safety is 12.

## 10. Lemonade Stand

Suppose I run a lemonade stand, which costs me \$100 a day to operate. I sell a drink of lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, $n_1$ people walk by my stand, and each buys a drink independently with probability $p_1$. If it isn't raining, $n_2$ people walk by my stand, and each buys a drink independently with probability $p_2$. It rains each day with probability $p_3$, independently of every other day. Let $X$ be my profit over the next week. In terms of $n_1, n_2, p_1, p_2$ and $p_3$, what is $\mathbb{E}[X]$?

**Solution:**

Let $R$ be the event it rains. Let $X_i$ be how many drinks I sell on day $i$ for $i = 1, \ldots, 7$. We are interested in $X = \sum_{i=1}^{7}(20X_i - 100)$. We have $X_i|R \sim \text{Binomial}(n_1, p_1)$, so $\mathbb{E}[X_i|R] = n_1p_1$. Similarly, $X_i|R^C \sim \text{Binomial}(n_2, p_2)$, so $\mathbb{E}\left[X_i|R^C\right] = n_2p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i|R]\,\mathbb{P}(R) + \mathbb{E}\left[X_i|R^C\right]\mathbb{P}(R^C) = n_1p_1p_3 + n_2p_2(1-p_3)$$

Hence, by linearity of expectation,

$$\begin{aligned}\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7}(20X_i - 100)\right] &= 20\sum_{i=1}^{7}\mathbb{E}[X_i] - 700 = 140\mu - 700 \\ &= 140 \cdot (n_1p_1p_3 + n_2p_2(1-p_3)) - 700.\end{aligned}$$

## 11. 3 points on a line

Three points $X_1, X_2, X_3$ are selected at random on a line $L$ (continuous independent uniform distributions). What is the probability that $X_2$ lies between $X_1$ and $X_3$? **Solution:**

Let $X_1, X_2, X_3 \sim Unif(0,1)$.

$$\begin{aligned}\mathbb{P}(X_1 < X_2 < X_3) &= \int_{-\infty}^{\infty}\mathbb{P}(X_1 < X_2 < X_3 \mid X_2 = x)\, f_{X_2}(x)\,dx && \text{Continuous LoTP} \\ &= \int_{-\infty}^{\infty}\mathbb{P}(X_1 < x, X_3 > x)\, f_{X_2}(x)\,dx && \text{Independence of } X_1, X_2, X_3 \\ &= \int_{-\infty}^{\infty}\mathbb{P}(X_1 < x)\,\mathbb{P}(x < X_3)\, f_{X_2}(x)\,dx && \text{Independence of } X_1, X_3 \\ &= \int_{-\infty}^{\infty}F_{X_1}(x)\,(1 - F_{X_3}(x))\, f_{X_2}(x)\,dx \\ &= \int_0^1 x\,(1-x)\,1\,dx \\ &= \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}\end{aligned}$$