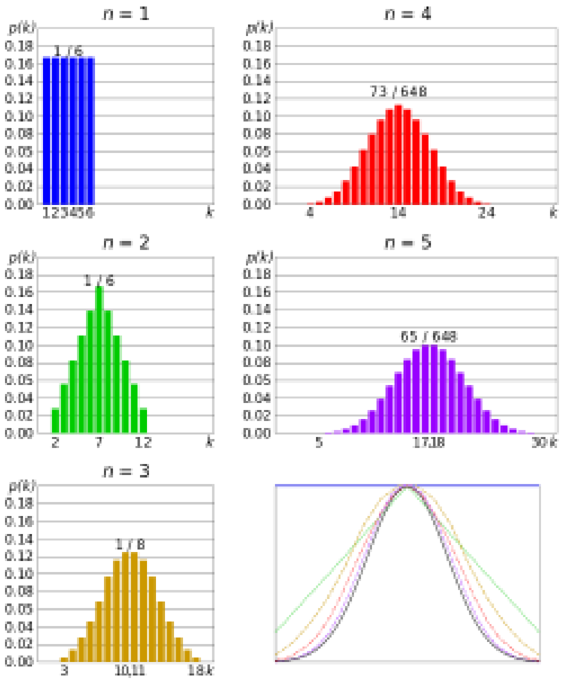
CSE 312

Section 7: Continuous RVs and CLT



Central Limit Theorem

Slides made by Harrison Bay

Administrivia

- Midterm grades are out
- HW5 was due on Wednesday (5/8)
- HW6 released and due next Wednesday (5/15)

- Check how many late days you have used (8 late days in total) and reach out if you have any concerns

If $Z \sim N(0,1)$ $---$ $P(Z \leq 0.42) = \Phi(0.42) = 0.66276$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189

Problem 3 Solution

- (a) Let X be a normal random with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute $P(4 < X < 16)$.

  Let $\frac{X-10}{6} = Z$. (Recall standardization: $Z \sim N(0,1)$)

  $$P(4 < X < 16) = P\left(\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\right) = P(-1 < Z < 1) = \Phi(1) - \Phi(-1) \approx 0.68268$$

- (b) Let X be a normal random variable with mean 5. If $P(X > 9) = 0.2$, approximately what is $Var(X)$?

  Let $\sigma^2 = \mathrm{Var}(X)$. Then $P(X > 9) = P\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right) = 1 - \Phi\left(\frac{4}{\sigma}\right) = 0.2$

  From the phi table, we get that $\frac{4}{\sigma} = 0.845$. Therefore, $\sigma \approx 4.73 \Rightarrow \sigma^2 \approx 22.4$.

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Problem 4b

- (b) Lots of statistics (like standardized test scores or heights) use percentiles to give context to where outcomes fall in a distribution. The nth percentile marks the outcome at which n% of the data points are less than the outcome. Let Y be a normal random variable with parameters $\mu = 15$ and $\sigma^2 = 4$. What value y marks the 85th percentile? What value b marks the 15th percentile?

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We first find y, which marks the 85th percentile, so $P(Y \leq y) = 0.85$. Let $\frac{Y-15}{2} = Z$. By the scale and shift properties of normal random variables, $Z \sim N(0, 1)$. Thus, we must find z such that $P(Z \leq z) = 0.85$.

$$\Phi(z) = P(Z \leq z) = 0.85$$
$$\Phi^{-1}\left(\Phi(z)\right) = \Phi^{-1}(0.85)$$

Thus, $z \approx 1.04$ by looking up the phi values in reverse to undo the Φ function.

Then $\frac{y-15}{2} = z \approx 1.04, so\ y \approx 17.08.$

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Recall that normal distributions are symmetric around the mean, where P(Y ≤ μ) = 0.5. Since

$$|P(Y \leq \mu) - P(Y \leq y)| = |0.5 - 0.85| = 0.35 = |P(Y \leq \mu) - P(Y \leq b)|,$$
$$b = \mu - |b - \mu| = 15 - |17.08 - 15| = 12.92,$$

so $b \approx 12.92.$

Problem 7

- A prolific Twitter user tweets approximately 350 tweets per week. Let’s assume for simplicity that the tweets are independent and each consists of a uniformly random number of characters between 10 and 140. Thus, the CLT implies that the number of characters tweeted by this user is approximately normal. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week.
  - Use continuity correction

Problem 7 solution

- Let X be the total number of characters tweeted by a twitter user in a week. Let $X_i \sim Unif(10, 140)$ be the number of characters in the ith tweet (since the start of the week).

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  - X is the sum of 350 IID RVs with $\mu = 75$ and $\sigma^2 = 1430$ (recall that $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a+1)^2-1}{12}$ for the discrete uniform RV), so $X \approx N \sim Normal(350 \cdot 75, 350 \cdot 1430)$.

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- We find: $P(26{,}000 \leq X \leq 27{,}000) = P(25{,}999.5 \leq X \leq 27{,}000.5)$
  - Notice the continuity correction is an equality, *not* an approximation

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  - Apply CLT: $P(25{,}999.5 \leq X \leq 27{,}000.5) \approx P(25{,}999.5 \leq N \leq 27{,}000.5)$
  - Standardize: $P(25{,}999.5 \leq N \leq 27{,}000.5) = P\left(\frac{25999.5-350\cdot 75}{\sqrt{350\cdot 1430}} \leq \frac{N-350\cdot 75}{\sqrt{350\cdot 1430}} \leq \frac{27000.5-350\cdot 75}{\sqrt{350\cdot 1430}}\right)$
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  - $\approx P\left(-0.3541 \leq \frac{N-350\cdot75}{\sqrt{350\cdot1430}} \leq 1.0608\right)$ (approximation due to <u>rounding</u>)
  - This is $= \Phi(1.0608) - \Phi(-0.3541) \approx 0.4923$ (equals, then approximation due to rounding in Φ table)

Problem 12

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

(a) Identify the range of X (ΩX), the range of Y (ΩY), and their joint range (ΩX,Y).

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

(c) Find the marginal PMF for Y , $p_Y(y)$ for $y \in \Omega_Y$.

(d) Are X and Y independent? Why or why not?

(e) Find $E[X^3 Y]$

Work on this with the people around you and then we'll go over it together!

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the values in the table show the joint probabilities - for example this highlighted value is P(X=1, Y-3)

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(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

- Based on the table (specifically what's highlighted in yellow and pink), we know that the range of X is {0, 1}, and the range of Y is {1,2,3}

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- The joint range is the set of *pairs of x and y* that both X and Y can take on at the same time. In other word the pairs of values x and y such that the joint PMF P(X=x, Y=y) is greater than 0.

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- In this case, that would be the set of pairs:

  $\Omega_{X,Y}$ = {(0, 2), (0,3), (1,1), (1,3)}
- We ***don't*** include the pairs (0, 1) and (1,2) because the joint pmf is 0 for those pairs

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- The marginal PMF for X is just like asking what the PMF for X is based on the joint PMF
- There are only two values in the range of X so we can try defining this PMF by looking at the probabilities for each value separately!
- First, we want to find P(X=0) -

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P(X=0) = P(X=0, Y=1)+P(X=0,Y=2) + P(X=0,Y=3)=0+0.2+0.1-0,3

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- Similarly, P(X=1) = 0.3+0+0.4

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- Similarly, P(X=1) = 0.3+0+0.4
- <u>In general, the marginal pmf looks like:</u>

$$p_X(x) = \sum_{y \in \Omega_Y} p_{X,Y}(x, y))$$

Suppose X and Y have the following joint PMF:

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(c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

$$p_Y(1) = \sum_x p_{X,Y}(x,1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x,2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x,3) = 0.1 + 0.4 = 0.5$$

Suppose X and Y have the following joint PMF:

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1	0.3	0	0.4

(d) Are X and Y independent? Why or why not?

- If we want to prove that X and Y <u>are</u> independent, we would need to show <u>both</u> of the conditions to be true:

$$\forall x, y, p_{X,Y}(x,y) = p_X(x)p_Y(y)$$
$$\Omega_{X,Y} = \Omega_X \times \Omega_Y$$

- However, note that the second condition is not true in this case! (take a look at the ranges we found in part a). So, X and Y are not independent.

Problem 5

Let X be the sum of 100 real numbers, and let Y be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that $|X - Y| > 3$?

Work on this with the people around you and then we'll go over it together!

Problem 5

- Notation: $X = \sum_{i=1}^{100} X_i$ and $Y = \sum_{i=1}^{100} r(X_i)$, where $r(\cdot)$ rounds to the nearest integer. Then

$$X - Y = \sum_{i=1}^{100} X_i - r(X_i)$$

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- Each $X_i - r(X_i)$ is round off error which is distributed as $Uni(-0.5, 0.5)$. We know the expectation and variance of a continuous uniform distribution: $\mu = 0$ and $\sigma^2 = \frac{1}{12}$

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- Each $X_i - r(X_i)$ is round off error which is distributed as $Uni(-0.5, 0.5)$. We know the expectation and variance of a continuous uniform distribution: $\mu = 0$ and $\sigma^2 = \frac{1}{12}$
- We then use the CLT: $X - Y \approx W \sim N\left(0, \frac{100}{12}\right)$
- Standardize:
$$P(|X - Y| > 3) \approx P(|W| > 3) = 2P(W > 3) \approx 2P(Z > 1.039)$$
$$\approx 0.29834$$

Problem 11

- Let $X_1, X_2, \ldots, X_n$ be IID RVs with CDF $F_X(x)$ and PDF $f_X(x)$. Let $Y = \min(X_1, \ldots, X_n)$ and let $Z = \max(X_1, \ldots, X_n)$. Show how to write the CDF and PDF of Y and Z in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

Work on this with the people around you and then we'll go over it together!

Problem 11 Solution

- Compute the CDFs of Z and Y (chain of equivalences):

$$F_Z(z) = P(Z < z) = P(X_1 < z, \dots, X_n < z)$$
$$= P(X_1 < z) \cdot \dots \cdot P(X_n < z) = \left(F_X(z)\right)^n$$

$$F_Y(y) = P(Y < y) = 1 - P(Y > y) = 1 - P(X_1 > y, \dots, X_n > y)$$
$$= 1 - P(X_1 > y) \cdot \dots \cdot P(X_n > y) = 1 - \left(1 - F_X(y)\right)^n$$

Note that the second to last step follows due to IID-ness of X_i's.

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Note that the second to last step follows due to IID-ness of X_i's.

- Now compute the PDFs of Z and Y:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz}\left(F_X(z)\right)^n$$

$$= n \cdot F_X(z)^{n-1} \cdot \frac{d}{dz} F_X(z) = n \cdot F_X(z)^{n-1} \cdot f_X(z)$$

By similar logic, $f_Y(y) = \frac{d}{dy} F_Y(y) = n \cdot \left(1 - F_X(y)\right)^{n-1} \cdot f_X(y)$

Problem 9

- A point is chosen at random on a line segment of length L. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

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- Let X be the distance from the leftmost side of the stick to the random point. Then $X \sim Uni(0, L)$.

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- For the ratio to be less than $\frac{1}{4}$, the shorter segment has to be less than $\frac{L}{5}$ in length; in other words, $X < \frac{L}{5}$ or $X > \frac{4L}{5}$.
- We then use the CDF of continuous uniform distributions:

$$P\left(X \leq \frac{L}{5}\right) + P\left(X > \frac{4L}{5}\right) = F_X\left(\frac{L}{5}\right) + \left(1 - F_X\left(\frac{4L}{5}\right)\right)$$

$$= \frac{\frac{L}{5} - 0}{L - 0} + \left(1 - \frac{\frac{4L}{5} - 0}{L - 0}\right) = \frac{1}{5} + \left(1 - \frac{4}{5}\right) = \frac{2}{5}$$