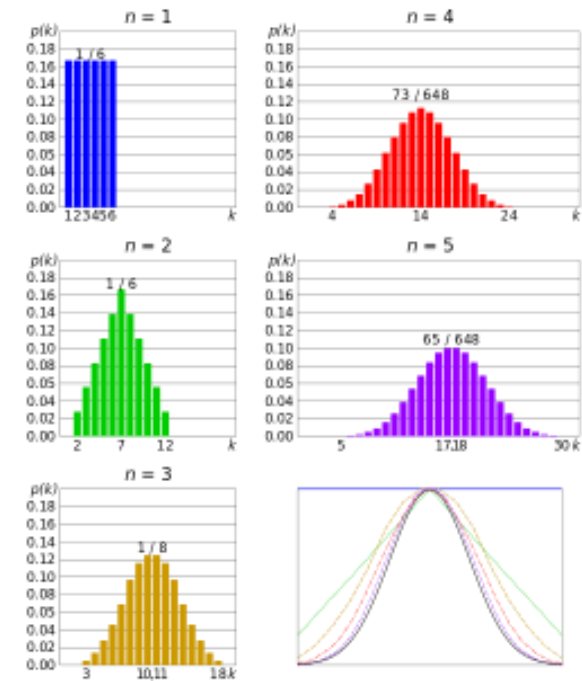


# CSE 312

## Section 7: Continuous RVs and CLT



Central Limit Theorem

# Administrivia

- Midterm grades are out
- HW5 was due on Wednesday (5/8)
- HW6 released and due next Wednesday (5/15)
  
- Check how many late days you have used (8 late days in total) and reach out if you have any concerns

$$\text{If } Z \sim N(0,1) \text{ --- } P(Z \leq 0.42) = \Phi(0.42) = 0.66276$$

| $z$ | 0.00    | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.06    | 0.07    | 0.08    | 0.09    |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.0 | 0.5     | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.5279  | 0.53188 | 0.53586 |
| 0.1 | 0.53983 | 0.5438  | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| 0.2 | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| 0.3 | 0.61791 | 0.62172 | 0.62552 | 0.6293  | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| 0.4 | 0.65542 | 0.6591  | 0.66276 | 0.6664  | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| 0.5 | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.7054  | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.7224  |
| 0.6 | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.7549  |
| 0.7 | 0.75804 | 0.76115 | 0.76424 | 0.7673  | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.7823  | 0.78524 |
| 0.8 | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| 0.9 | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| 1.0 | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| 1.1 | 0.86433 | 0.8665  | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.879   | 0.881   | 0.88298 |
| 1.2 | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| 1.3 | 0.9032  | 0.9049  | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92072 | 0.9222  | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |

# Problem 3

- (a) Let  $X$  be a normal random with parameters  $\mu = 10$  and  $\sigma^2 = 36$ . Compute  $P(4 < X < 16)$ .

Work on this with the people around you and then we'll go over it together!

- (b) Let  $X$  be a normal random variable with mean 5. If  $P(X > 9) = 0.2$ , approximately what is  $Var(X)$ ?

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# Problem 4b

- (b) Lots of statistics (like standardized test scores or heights) use percentiles to give context to where outcomes fall in a distribution. The  $n$ th percentile marks the outcome at which  $n\%$  of the data points are less than the outcome. Let  $Y$  be a normal random variable with parameters  $\mu = 15$  and  $\sigma^2 = 4$ . What value  $y$  marks the 85th percentile? What value  $b$  marks the 15th percentile?

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# Problem 7

- A prolific Twitter user tweets approximately 350 tweets per week. Let's assume for simplicity that the tweets are independent and each consists of a uniformly random number of characters between 10 and 140. Thus, the CLT implies that the number of characters tweeted by this user is approximately normal. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week.
  - Use continuity correction



# Problem 7 solution

# Problem 12

Suppose  $X$  and  $Y$  have the following joint PMF:

| $X/Y$ | 1   | 2   | 3   |
|-------|-----|-----|-----|
| 0     | 0   | 0.2 | 0.1 |
| 1     | 0.3 | 0   | 0.4 |

- (a) Identify the range of  $X$  ( $\Omega_X$ ), the range of  $Y$  ( $\Omega_Y$ ), and their joint range ( $\Omega_{X,Y}$ ).
- (b) Find the marginal PMF for  $X$ ,  $p_X(x)$  for  $x \in \Omega_X$ .
- (c) Find the marginal PMF for  $Y$ ,  $p_Y(y)$  for  $y \in \Omega_Y$ .
- (d) Are  $X$  and  $Y$  independent? Why or why not?
- (e) Find  $E[X^3Y]$

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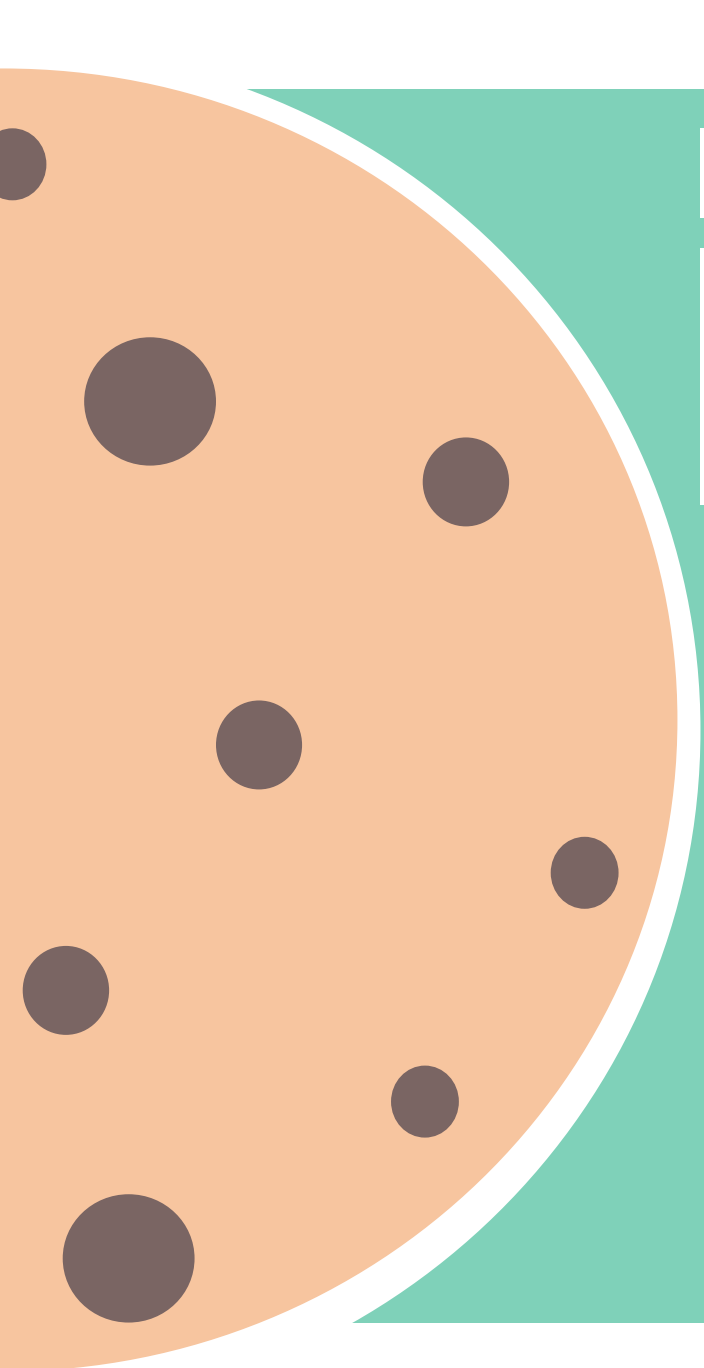
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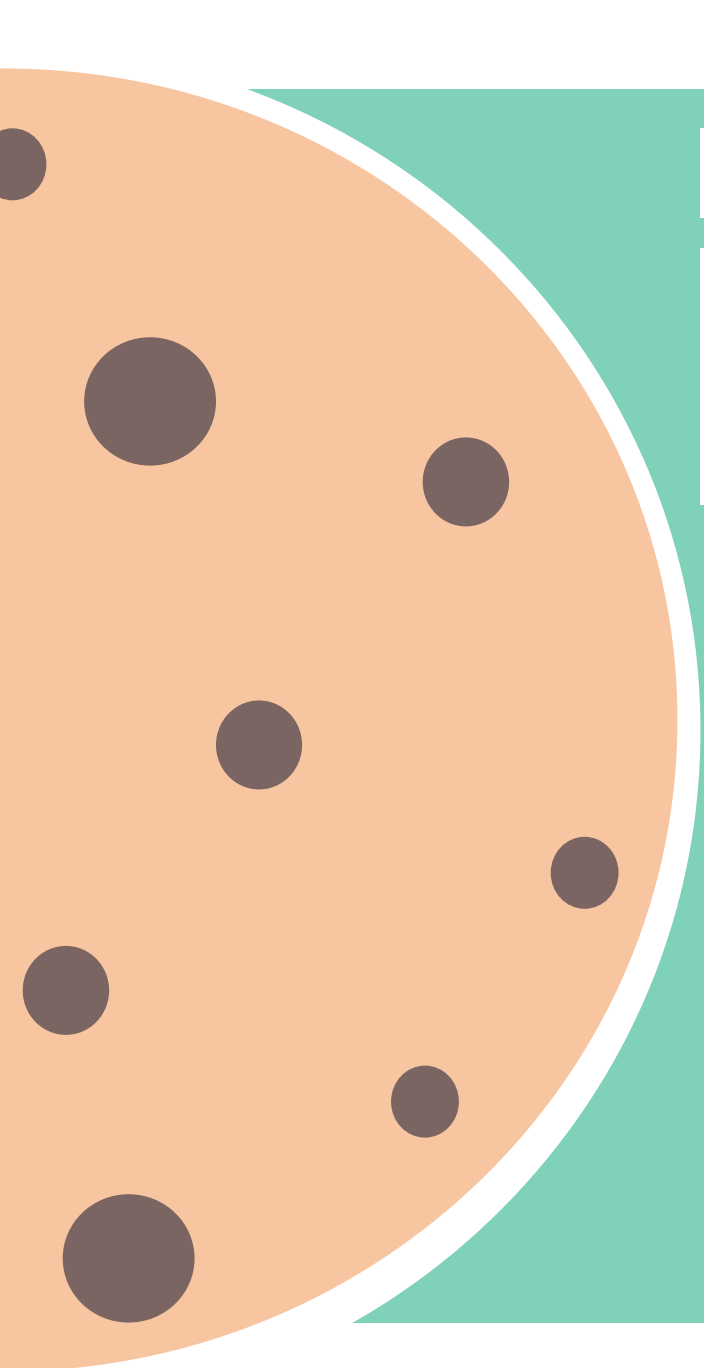


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these are  
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these are  
the values  
 $X$  takes on

the values in the table show the  
joint probabilities - for example this  
highlighted value is  $P(X=1, Y=3)$

Suppose  $X$  and  $Y$  have the following joint PMF:

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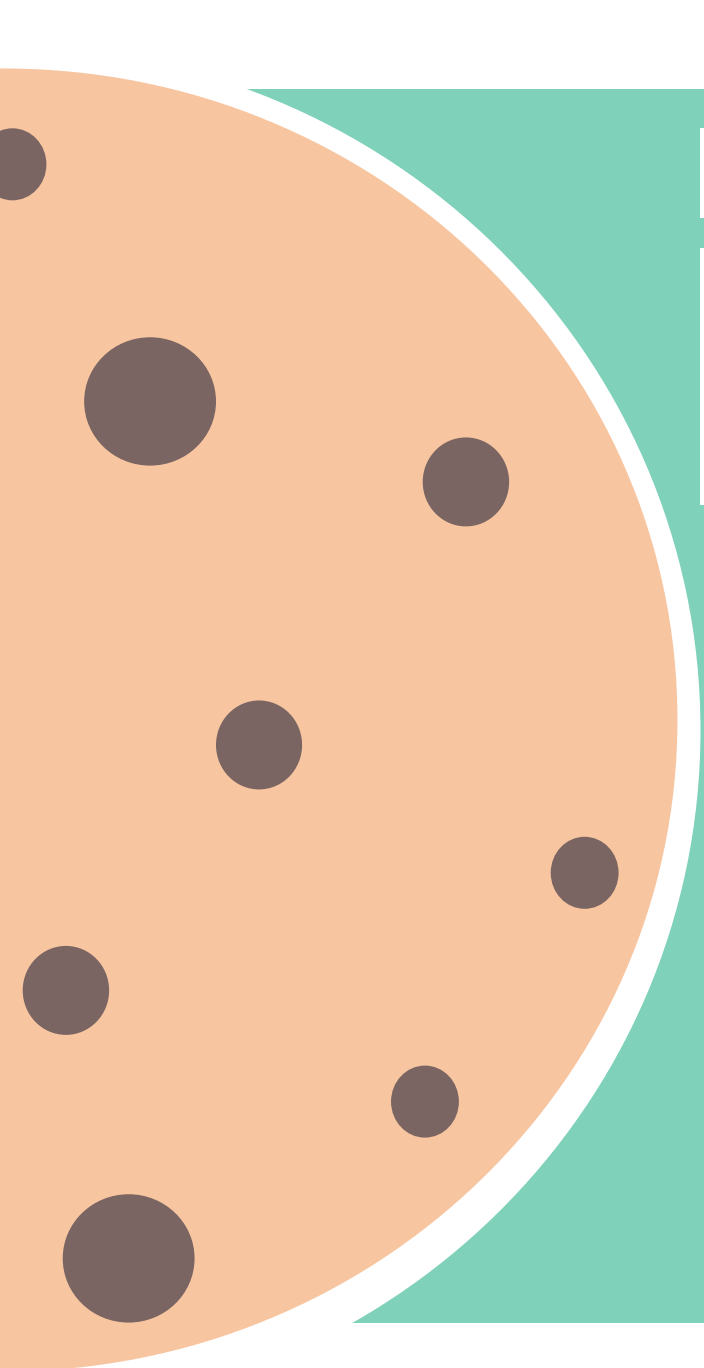
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- In this case, that would be the set of pairs:  
$$\Omega_{X,Y} = \{(0, 2), (0,3), (1,1), (1,3)\}$$
- We don't include the pairs  $(0, 1)$  and  $(1,2)$  because the joint pmf is 0 for those pairs



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(b) Find the marginal PMF for  $X$ ,  $p_X(x)$  for  $x \in \Omega_X$ .

- The marginal PMF for  $X$  is just like asking what the PMF for  $X$  is based on the joint PMF
- There are only two values in the range of  $X$  so we can try defining this PMF by looking at the probabilities for each value separately!
- First, we want to find  $P(X=0)$  -

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- First, we want to find  $P(X=0)$  - Based on the joint PMF, and using LOTP

$$P(X=0) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) = 0 + 0.2 + 0.1 = 0.3$$

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- Similarly,  $P(X=1) = 0.3 + 0 + 0.4$
- In general, the marginal pmf looks like:

$$p_X(x) = \sum_{y \in \Omega_Y} p_{X,Y}(x, y)$$

Suppose  $X$  and  $Y$  have the following joint PMF:

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(c) Find the marginal PMF for  $Y$ ,  $p_Y(y)$  for  $y \in \Omega_Y$ .

$$p_Y(1) = \sum_x p_{X,Y}(x, 1) = 0 + 0.3 = 0.3$$

$$p_Y(2) = \sum_x p_{X,Y}(x, 2) = 0.2 + 0 = 0.2$$

$$p_Y(3) = \sum_x p_{X,Y}(x, 3) = 0.1 + 0.4 = 0.5$$

Suppose  $X$  and  $Y$  have the following joint PMF:

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(d) Are  $X$  and  $Y$  independent? Why or why not?

- If we want to prove that  $X$  and  $Y$  are independent, we would need to show both of the conditions to be true:

$$\forall x, y, p_{X,Y}(x, y) = p_X(x)p_Y(y)$$
$$\Omega_{X,Y} = \Omega_X \times \Omega_Y$$

- However, note that the second condition is not true in this case! (take a look at the ranges we found in part a). So,  $X$  and  $Y$  are not independent.

# Problem 5

Let  $X$  be the sum of 100 real numbers, and let  $Y$  be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5, what is the approximate probability that  $|X - Y| > 3$ ?

Work on this with the people around you and then we'll go over it together!

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# Problem 11

- Let  $X_1, X_2, \dots, X_n$  be IID RVs with CDF  $F_X(x)$  and PDF  $f_X(x)$ . Let  $Y = \min(X_1, \dots, X_n)$  and let  $Z = \max(X_1, \dots, X_n)$ . Show how to write the CDF and PDF of  $Y$  and  $Z$  in terms of the functions  $F_X(\cdot)$  and  $f_X(\cdot)$ .

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# Problem 9

- A point is chosen at random on a line segment of length  $L$ . Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than  $\frac{1}{4}$ .