Section 6: Continuous Random Variables

Review of Main Concepts

- **Continuous Random Variable**: A continuous random variable X is one for which its cumulative distribution function $F_X(x) : \mathbb{R} \to \mathbb{R}$ is continuous everywhere. A continuous random variable has an uncountably infinite number of values.
- **Cumulative Distribution Function (cdf)**: For any random variable (discrete or continuous) X, the cumulative distribution function is defined as $F_X(x) = \mathbb{P}(X \leq x)$. Notice that this function must be monotonically nondecreasing: if $x < y$ then $F_X(x) \leq F_X(y)$, because $\mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y)$. Also notice that since probabilities are between 0 and 1, that $0 \leq F_X(x) \leq 1$ for all x, with $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to+\infty} F_X(x) = 1$. Also, since $P(X = k) = 0$ for some constant k if X is a continuous random variable, $P(X < k) = P(X \leq k)$
- **Probability Density Function (pdf or density)**: Let X be a continuous random variable. Then the probability density function $f_X(x) : \mathbb{R} \to \mathbb{R}$ of X is defined as $f_X(x) = \frac{d}{dx}F_X(x)$. Turning this around, it means that $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$. From this, it follows that $\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$ and that $\int_{-\infty}^{\infty} f_X(x)dx = 1$. From the fact that $F_X(x)$ is monotonically nondecreasing it follows that $f_X(x) \geq 0$ for every real number x.

  If X is a continuous random variable, note that in general $f_X(a) \neq \mathbb{P}(X = a)$, since $\mathbb{P}(X = a) = F_X(a) - F_X(a) = 0$ for all a. However, the probability that X is close to a is proportional to $f_X(a)$: for small δ, $\mathbb{P}\left(a - \frac{\delta}{2} < X < a + \frac{\delta}{2}\right) \approx \delta f_X(a)$.
- **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) if they are independent and have the same probability mass function or probability density function.
- **Discrete to Continuous:** To summarize, when going from discrete to continuous, the main differences are usually - using an integral instead of a summation, and using the density function $f_X(k)$ instead of the PMF $P(X = k)$.

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
Expectation	$\mathbb{E}[X] = \sum_x x p_X(x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

- **Law of Total Probability (Continuous): This has not been covered in class yet, but will be soon.** A is an event, and X is a continuous random variable with density function $f_X(x)$.

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X = x) f_X(x)dx$$

- **Zoo of *Continuous* Random Variables**
  (a) **Uniform**: $X \sim \mathsf{Uniform}(a, b)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

  $\mathbb{E}[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$. This represents each real number from $[a, b]$ to be equally likely.

  (b) **Exponential**: $X \sim \mathsf{Exponential}(\lambda)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$. $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. The exponential random variable is the continuous analog of the geometric random variable: it represents the waiting time to the next event, where $\lambda > 0$ is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential random variable X is memoryless:

$$\text{for any } s, t \geq 0,\ \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

The geometric random variable also has this property.

1. Content Review

(a) What is $P(X = 4)$ if X is a **continuous** random variable?

◯ 1

◯ 0

◯ not enough information

(b) The cumulative distribution function for a continuous random variable X is $F_X(k) =$

◯ $\int_{-\infty}^{k} f_X(x)dx$

◯ $\int_{-\infty}^{\infty} f_X(x)dx$

◯ $\int_{k}^{\infty} f_X(x)dx$

◯ $\frac{d}{dk} f_X(k)$

(c) The probability density function for a continuous random variable X is $f_X(k) =$

◯ $\int_{-\infty}^{k} f_X(x)dx$

◯ $\frac{d}{dk} F_X(k)$

(d) **True** or **False**. If X is a continuous random variable, $E[X] = \int_{\infty}^{\infty} x f_X(x)dx$

(e) **True** or **False**. If X is a continuous random variable, $Var(X) = E[X^2] - (E[X])^2$

(f) Which of the following follow an Exponential(λ) distribution?

◯ Number of minutes to the first success with λ as average number of successes per minute

◯ Number of successes in the first 1 minute with λ as average number of successes per minute

◯ Time (real number) to the first success with λ as average number of successes per minute

2. Uniform2

Robbie decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim \text{Uniform2}(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

3. Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but

$$\mathbb{P}(1 \le X \le 2) = 2 \cdot \mathbb{P}(0 \le X < 1).$$

Outside of $[0, 2]$ the density is 0. What is the PDF and CDF of X?

4. Max of uniforms

Let $U_1, U_2, \ldots, U_n$ be mutually independent Uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \ldots, U_n)$.

5. New PDF?

Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f_X(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

6. Throwing a dart

Consider the closed unit circle of radius r, i.e., $S = \{(x, y) : x^2 + y^2 \leq r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely this means that the probability that the dart lands in any particular area of size A (that is entirely inside the circle of radius R), is equal to $\frac{\text{A}}{\text{Area of whole circle}}$. The density outside the circle of radius r is 0.

Let X be the distance the dart lands from the center. What is the CDF and pdf of X? What is $\mathbb{E}[X]$ and $Var(X)$?

7. A square dartboard?

You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest *square* B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For X, find the CDF, PDF, $\mathbb{E}[X]$, and $Var(X)$.

8. Will the battery last?

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with expectation 10,000 miles. If the owner wants to take a 5000 mile road trip, what is the probability that she will be able to complete the trip without replacing the battery, given that the car has already been used for 2000 miles?

9. Batteries and exponential distributions

Let X_1, X_2 be independent exponential random variables, where X_i has parameter λ_i, for $1 \leq i \leq 2$. Let $Y = \min(X_1, X_2)$.

(a) Show that Y is an exponential random variable with parameter $\lambda = \lambda_1 + \lambda_2$. Hint: Start by computing $\mathbb{P}(Y > y)$. Two random variables with the same CDF have the same pdf. Why?

(b) What is $Pr(X_1 < X_2)$? (Use the law of total probability.) The law of total probability hasn't been covered in class yet, but will be soon at which point it would be good to revisit this problem!

(c) You have a digital camera that requires two batteries to operate. You purchase n batteries, labelled $1, 2, \ldots, n$, each of which has a lifetime that is exponentially distributed with parameter λ, independently of all other batteries. Initially, you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process, you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer.

(d) In the scenario of the previous part, what is the probability that battery i is the last remaining battery as a function of i? (You might want to use the memoryless property of the exponential distribution that has been discussed.)

10. Continuous Law of Total Probability?

This has not been covered in class yet, but will be soon.

In this exercise, we will extend the law of total probability to the continuous case.

(a) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n+1$). Let H be the event that the coin comes up heads. What is $\mathbb{P}(H)$?

(b) Now suppose $U \sim \mathsf{Uniform(0,1)}$ has the *continuous* uniform distribution over the interval $[0, 1]$. Extend the law of total probability to work for this continuous case. (Hint: you may have an integral in your answer instead of a sum).

(c) Let's generalize the previous result we just used. Suppose E is an event, and X is a continuous random variable with density function $f_X(x)$. Write an expression for $\mathbb{P}(E)$, conditioning on X.

11. Transformations

This has not been covered in class yet and probably won't be. But if you're interested, please read Section 4.4 from the textbook.

Suppose $X \sim \mathsf{Uniform}(0, 1)$ has the continuous uniform distribution on $(0, 1)$. Let $Y = -\frac{1}{\lambda} \log X$ for some $\lambda > 0$.

(a) What is Ω_Y?

(b) First write down $F_X(x)$ for $x \in (0, 1)$. Then, find $F_Y(y)$ on Ω_Y.

(c) Now find $f_Y(y)$ on Ω_Y (by differentiating $F_Y(y)$ with respect to y. What distribution does Y have?

12. Convolutions

This has not been covered in class. We're not yet sure if we will have time for it, but if you're interested, please read Section 5.5 from the textbook.

Suppose $Z = X + Y$, where $X \perp Y$. ($\perp$ is the symbol for independence. In other words, X and Y are independent.) Z is called the convolution of two random variables. If X, Y, Z are discrete,

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

If X, Y, Z are continuous,

$$F_Z(z) = \mathbb{P}(X + Y \leq z) = \int_{-\infty}^{\infty} \mathbb{P}(Y \leq z - X | X = x) f_X(x) dx = \int_{-\infty}^{\infty} F_Y(z - x) f_X(x) dx$$

Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

(a) Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using a similar idea to convolution, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

(b) Find s, where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.