CSE 312 Section 6

Continuous RVs

Administrivia

Announcements & Reminders

- Midterm Exam
  - Don’t discuss the midterm

- HW5
  - Released last night
  - Written is due Wednesday 5/08 @ 11:59pm
  - Late deadline Saturday 5/11 @ 11:59pm

Review & Questions

Any lingering questions from this last week?

Each week in section, we’ll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we’ll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Kahoot for content review!

see task 1 from section handout

Review of Main Concepts

- **Cumulative Distribution Function (CDF)**: For any random variable X, the CDF is defined as $F_X(x) = \mathbb{P}(X \leq x)$.
  - Notice the CDF is monotonic (non-decreasing), i.e. if $x < y$ then $F_X(x) \leq F_X(y)$
  - The CDF is bounded between 0 and 1.
- A **Continuous RV** is one for which the CDF is continuous everywhere.
  - Support is an uncountably infinite set.
- **Probability Density Function (PDF)** is defined as $f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x)$
  - Implies that $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t$
  - $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(t)\mathrm{d}t$
  - $f_X(x) \geq 0$

Review of Main Concepts

- **I.I.D**: Random variables are "independent and identically distributed" if they are independent and have the same PDF/PMF
- For continuous random variables:
  - $\mathbb{P}(X = x) = 0 \neq f_X(x)$
  - $F_X(x) = \int_{-\infty}^{x} f_X(t)\mathrm{d}t$
  - $\int_{-\infty}^{\infty} f_X(t)\mathrm{d}t = 1$
  - $\mathbb{E}[X] = \int_{-\infty}^{\infty} t \cdot f_X(t)\mathrm{d}t$
  - $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(t) \cdot f_X(t)\mathrm{d}t$

Problem 2 – Uniform2

2 – Uniform2

Robbie decided he wanted to create a “new” type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We’ll denote a random variable X having the “Uniform-2” distribution as $X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \ \infty)$. (Hint: use a piecewise definition).

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \ \infty)$. (Hint: use a piecewise definition).

Work on this problem with the people around you, and then we’ll go over it together!

2 – Uniform2

$X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

2 – Uniform2

$X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

$$f_X(x) = \begin{cases} \dfrac{1}{(b - a) + (d - c)}, & x \in [a, b] \cup [c, d] \\ 0, & otherwise \end{cases}$$

2 – Uniform2

$X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty,\ \infty)$. (Hint: use a piecewise definition).

2 – Uniform2

$X \sim Uniform2(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \ \infty)$. (Hint: use a piecewise definition).

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, a) \\ \dfrac{(x - a)}{(b - a) + (d - c)}, & x \in [a, b) \\ \dfrac{(b - a)}{(b - a) + (d - c)}, & x \in [b, c) \\ \dfrac{(b - a) + (x - c)}{(b - a) + (d - c)}, & x \in [c, d) \\ 1, & x \in [d, \infty) \end{cases}$$

Problem 6 – Throwing a dart

6 – Throwing a dart

Consider the closed unit circle of radius r, i.e. S={ $(x, y) : x^2 + y^2 \leq r^2$ }. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely, this means that the probability that the dart lands in any particular area of size A is equal to $\frac{A}{Area\ of\ the\ whole\ circle}$. The density outside the unit circle is 0.

Let X be the distance the dart lands from the center. What is the CDF and PDF of X? What is $E[X]$ and $Var(X)$?

Work on this problem with the people around you, and then we'll go over it together!

6 – Throwing a dart

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\frac{\pi \cdot x^2}{\pi \cdot r^2}$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/r^2 & 0 \leq x \leq r \\ 1 & x > r \end{cases}$$

6 – Throwing a dart

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\frac{\pi \cdot x^2}{\pi \cdot r^2}$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/r^2 & 0 \leq x \leq r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) \begin{cases} 2x/r^2 & 0 < x \leq r \\ 0 & otherwise \end{cases}$$

6 – Throwing a dart

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\frac{\pi \cdot x^2}{\pi \cdot r^2}$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/r^2 & 0 \le x \le r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) \begin{cases} 2x/r^2 & 0 < x \le r \\ 0 & otherwise \end{cases}$$

Using the definition of expectation we get:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^r \frac{2x^2}{r^2} dx = \frac{2}{3x^2}\left(x^3 \Big|_0^r\right) = \frac{2}{3} r$$

6 – Throwing a dart

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\frac{\pi \cdot x^2}{\pi \cdot r^2}$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/r^2 & 0 \leq x \leq r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) \begin{cases} 2x/r^2 & 0 < x \leq r \\ 0 & otherwise \end{cases}$$

Using the definition of expectation we get:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^r \frac{2x^2}{r^2} dx = \frac{2}{3x^2}\left(x^3 \Big|_0^r\right) = \frac{2}{3} r$$

We know that $Var(X) = E[X^2] - E[X]^2$:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^r \frac{2x^3}{r^2} dx = \frac{2}{4r^2}\left(x^4 \Big|_0^r\right) = \frac{1}{2} r^2$$

Plugging this into our variance equation gives:

$$\Rightarrow Var(X) = \frac{1}{2} r^2 - \left(\frac{2}{3} r\right)^2 = \frac{1}{18} r^2$$

Problem 3 – Create the distribution

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but

$$\mathbb{P}(1 \le X \le 2) = 2 \cdot \mathbb{P}(0 \le X \le 1)$$

Outside of $[0, 2]$, the density is 0. What is the PDF and CDF of X?

Work on this problem with the people around you, and then we'll go over it together!

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but $\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$. Outside of $[0, 2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but $\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$. Outside of $[0, 2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$.

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0,1)$ and uniform on $[1,2]$, but $\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$. Outside of $[0,2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$.

The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0,1)$ and uniform on $[1,2]$, but $\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$. Outside of $[0,2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$. The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

Additionally,

$$d = F_X(2) - F_X(1) = P(1 \leq X \leq 2) = 2P(0 \leq X \leq 1) = 2\big(F_X(1) - F_X(0)\big) = 2c$$

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1)$ and uniform on $[1, 2]$, but $\mathbb{P}(1 \le X \le 2) = 2 \cdot \mathbb{P}(0 \le X \le 1)$. Outside of $[0, 2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \le x < 1 \\ d, & 1 \le x \le 2 \\ 0, & otherwise \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$. The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

Additionally,

$$d = F_X(2) - F_X(1) = P(1 \le X \le 2) = 2P(0 \le X \le 1) = 2\big(F_X(1) - F_X(0)\big) = 2c$$

To solve for c and d in our PDF, we need only solve the system of two equations from above: $d = 2c, d + c = 1$ So, $d = \frac{2}{3}, c = \frac{1}{3}$.

3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0,1)$ and uniform on $[1,2]$, but $\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1)$. Outside of $[0,2]$, the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c, & 0 \leq x < 1 \\ d, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$. The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

Additionally,

$$d = F_X(2) - F_X(1) = P(1 \leq X \leq 2) = 2P(0 \leq X \leq 1) = 2\big(F_X(1) - F_X(0)\big) = 2c$$

To solve for c and d in our PDF, we need only solve the system of two equations from above: $d = 2c, d + c = 1$ So, $d = \frac{2}{3}, c = \frac{1}{3}$. Taking the integral of the PDF yields the CDF, which looks like

$$f_X(x) = \begin{cases} 1/3, & 0 \leq x < 1 \\ 2/3, & 1 \leq x \leq 2 \\ 0, & otherwise \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{3}, & 0 \leq x < 1 \\ \frac{2x}{3} - \frac{1}{3}, & 1 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

Problem 4 – Max of uniforms

4 – Max of uniforms

Let $U_1, U_2, \dots, U_n$ be mutually independent uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \dots, U_n)$.

Work on this problem with the people around you, and then we'll go over it together!

4 – Max of uniforms

Let $U_1, U_2, \dots, U_n$ be mutually independent uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \dots, U_n)$.

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \dots, a_n)$ is less than some constant c, iff each individual number is less than that constant c (i.e. $\mathrm{a_i} < c$ for all i). Using this idea, we get

4 – Max of uniforms

Let $U_1, U_2, \ldots, U_n$ be mutually independent uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \ldots, U_n)$.

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \ldots, a_n)$ is less than some constant c, iff each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

$$\begin{aligned} F_Z(x) = \mathbb{P}(Z \leq x) &= \mathbb{P}(\max(U_1, \ldots, U_n) \leq x) \\ &= \mathbb{P}(U_1 \leq x, \ldots, U_n \leq x) \\ &= \mathbb{P}(U_1 \leq x) \ldots \mathbb{P}(U_n \leq X) && \text{by independence} \\ &= F_{U_1}(x) \ldots F_{U_n}(x) \\ &= F_U(x)^n && \text{where } U \sim Unif(0,1) \end{aligned}$$

4 – Max of uniforms

Let $U_1, U_2, \dots, U_n$ be mutually independent uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \dots, U_n)$.

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \dots, a_n)$ is less than some constant c, iff each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

$$\begin{aligned} F_Z(x) = \mathbb{P}(Z \le x) &= \mathbb{P}(\max(U_1, \dots, U_n) \le x) \\ &= \mathbb{P}(U_1 \le x, \dots, U_n \le x) \\ &= \mathbb{P}(U_1 \le x) \dots \mathbb{P}(U_n \le X) && \text{by independence} \\ &= F_{U_1}(x) \dots F_{U_n}(x) \\ &= F_U(x)^n && \text{where } U \sim Unif(0,1) \end{aligned}$$

So, the CDF of Z is then

$$F_Z(x) = \begin{cases} 0, & x < 0 \\ x^n, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

4 – Max of uniforms

Let $U_1, U_2, \ldots, U_n$ be mutually independent uniform random variables on $(0, 1)$. Find the CDF and PDF for the random variable $Z = \max(U_1, \ldots, U_n)$.

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \ldots, a_n)$ is less than some constant c, iff each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

$$
\begin{aligned}
F_Z(x) = \mathbb{P}(Z \le x) &= \mathbb{P}(\max(U_1, \ldots, U_n) \le x) \\
&= \mathbb{P}(U_1 \le x, \ldots, U_n \le x) \\
&= \mathbb{P}(U_1 \le x) \ldots \mathbb{P}(U_n \le X) && \text{by independence} \\
&= F_{U_1}(x) \ldots F_{U_n}(x) \\
&= F_U(x)^n && \text{where } U \sim Unif(0,1)
\end{aligned}
$$

So, the CDF of Z is then

$$
F_Z(x) = \begin{cases} 0, & x < 0 \\ x^n, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}
$$

Finding the PDF is just taking the derivative:

$$
f_Z(x) = \begin{cases} n \cdot x^{n-1}, & 0 \le x \le 1 \\ 0, & otherwise \end{cases}
$$

That's All, Folks!

Thanks for coming to section this week!
Any questions?