Section 3: Solutions

Review of Main Concepts

- **Conditional Probability**: $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$
- **Independence**: Events E and F are independent iff $\Pr(E \cap F) = \Pr(E)\Pr(F)$, or equivalently $\Pr(F) = \Pr(F|E)$, or equivalently $\Pr(E) = \Pr(E|F)$
- **Bayes Theorem**: $\Pr(A|B) = \dfrac{\Pr(B|A)\Pr(A)}{\Pr(B)}$
- **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff
  - $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
  - $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
- **Law of Total Probability (LTP)**: Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then

  $\Pr(B) = \sum_{i=1}^{n} \Pr(B \cap A_i) = \sum_{i=1}^{n} \Pr(\text{B} \mid A_i)\Pr(A_i)$
- **Bayes Theorem with LTP**: Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then $\Pr(A_1|B) = \dfrac{\Pr(B \mid A_1)\Pr(A_1)}{\sum_{i=1}^{n} \Pr(\text{B} \mid A_i)\Pr(A_i)}$. In particular, $\Pr(A|B) = \dfrac{\Pr(B \mid A)\Pr(A)}{\Pr(\text{B} \mid A)\Pr(A) + \Pr(\text{B} \mid A^C)\Pr(A^C)}$
- **Chain Rule**: Suppose $A_1, ..., A_n$ are events. Then,

$$\Pr(A_1 \cap ... \cap A_n) = \Pr(A_1)\Pr(A_2|A_1)\Pr(A_3|A_1 \cap A_2) ... \Pr(A_n|A_1 \cap ... \cap A_{n-1})$$

Note: There are also content review slides posted on the website briefly going over these concepts!

1. Content Review

(a) True or False: It is always the case that $\Pr(A \mid B) = \Pr(B \mid A)$.

Solution:

False. It is not true in general that

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A \cap B)}{\Pr(A)} = \Pr(B \mid A)\ .$$

Finding events A and B such that $\Pr(A \cap B) > 0$ and $\Pr(A) \neq \Pr(B)$ would be a valid counterexample.

(b) Select one: Suppose A and B are independent events. Then

- ◯ $\Pr(A \cap B) = \Pr(A)\Pr(B)$
- ◯ $\Pr(A \mid B) = \Pr(A)$
- ◯ Both are true
- ◯ Both are false

Solution:

Both are true, as we have seen in lecture.

(c) Select one: For any two events A and B,

- ◯ $\Pr(A \mid B) = \Pr(A \cap B)$
- ◯ $\Pr(A \mid B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$
- ◯ $\Pr(A \mid B) = \frac{\Pr(B|A)\Pr(B)}{\Pr(A)}$

Solution:

$\Pr(A \mid B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}$ by definition of Bayes Theorem.

(d) True or False: Let A and B be the event that a six-sided die is at most 3 and at least 4, respectively. Then A and B are a partition.

Solution:

True. A and B are mutually exclusive – the die cannot roll both no more than 3 and more than 4 at the same time). Additionally, A and B are exhaustive: for any value $v \in \{1, 2, \ldots, 6\}$, we can assign v to one of A or B.

(e) Select one: Let T be the event that Alice tests positive for the cold. Let C be the event that Alice actually has the cold. Suppose the probability that Alice tests positive given that she has a cold is 0.8. Then the probability she tests negative given that she has a cold is

- ◯ 0.8
- ◯ 0.2
- ◯ Not enough information

Solution:

First, note that for two events A and B,

$$\Pr(A \mid B) + \Pr(\overline{A} \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} + \frac{\Pr(\overline{A} \cap B)}{\Pr(B)} = \frac{\Pr(A \cap B) + \Pr(\overline{A} \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1 \,,$$

where the first part uses the definition of conditional probability, and the last equality comes from using the Law of Total Probability in the numerator (since A and $\overline{A}$ partition the sample space).

Now, we are looking for $\Pr(\overline{T} \mid C)$. Then

$$\Pr(T \mid C) + \Pr(\overline{T} \mid C) = 1 \implies \Pr(\overline{T} \mid C) = 0.2 \,.$$

(f) True or False: Suppose $A_1, \ldots, A_n$ are events. Then

$$\Pr(A_1 \cap A_2 \cap \cdots \cap A_n) = \Pr(A_1) \cdot \Pr(A_2 \mid A_1) \Pr(A_3 \mid A_1 \cap A_2) \cdots \Pr(A_n \mid A_1 \cap \cdots \cap A_{n-1}) \,.$$

Solution:

True, by definition of the Chain Rule.

2. Random Grades?

Suppose there are three possible teachers for CSE 312: Martin Tompa, Anna Karlin, and Adam Blank. Suppose the probabilities of getting an A in Martin's class is $\frac{5}{15}$, for Anna's class is $\frac{3}{15}$, and for Adam's class is $\frac{1}{15}$. Suppose you are assigned a grade randomly according to the given probabilities when you take a class from one of these professors, irrespective of your performance. Furthermore, suppose Adam teaches your class with probability $\frac{1}{2}$ and Anna and Martin have an equal chance of teaching if Adam isn't. What is the probability you had Adam, given that you received an A? Compare this to the unconditional probability that you had Adam.

Solution:

Let T, K, B be the events you take 312 from Tompa, Karlin, and Blank, respectively. Let A be the event you get an A. We use Bayes' theorem with LTP, conditioning on each of T, K, B since those events partition our sample space.

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A|T)\Pr(T) + \Pr(A|K)\Pr(K) + \Pr(A|B)\Pr(B)} = \frac{1/15 \cdot 1/2}{5/15 \cdot 1/4 + 3/15 \cdot 1/4 + 1/15 \cdot 1/2}$$

$$= \frac{2}{5+3+2} = \boxed{\frac{1}{5}}$$

Note that we used Bayes' Theorem because we already know the reverse probability $Pr(A|B)$, which makes it easy for us to evaluate the initial probability $Pr(B|A)$.

3. Marbles in Pockets

Aleks has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If he transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

Solution:

Let $W\_, B\_$ denote the event that we choose a white marble or a blue marble respectively, with subscripts L, R indicating from which pocket we are picking – left and right, respectively.
We know that we will pick from the left pocket first, and right pocket second. We can then use the Law of Total Probability conditioning on the color of the transferred marble so that:

$$\Pr(B_R) = \Pr(W_L) \cdot \Pr(B_R|W_L) + \Pr(B_L) \cdot \Pr(B_R|B_L) = \frac{3}{8} \cdot \frac{4}{9} + \frac{5}{8} \cdot \frac{5}{9} = \boxed{\frac{37}{72}}$$

4. Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

(a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

Solution:

Let S_i be the event that she stayed during the i-th episode. Let B bet the event that should bribes the judges By the Law of Total Probability conditioning on whether the contestant bribed the judges we get,

$$\text{Pr}(S_1) = \text{Pr}(B)\ \text{Pr}(S_1 \mid B) + \text{Pr}(\overline{B})\ \text{Pr}(S_1 \mid \overline{B}) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} = \boxed{\frac{1}{2}}$$

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

Solution:

Let S_i be defined as before. Staying during both episodes is equivalent to the contestant staying in episodes 1 and 2, so the event $S_1 \cap S_2$. By the Law of Total Probability, we get:

$$\text{Pr}(S_1 \cap S_2) = \text{Pr}(B)\ \text{Pr}(S_1 \cap S_2 \mid B) + \text{Pr}(\overline{B})\ \text{Pr}(S_1 \cap S_2 \mid \overline{B}) \tag{1}$$

We know a contestant is guaranteed to stay on the show, given that they are bribing the judges, hence:

$$\text{Pr}(S_1 \cap S_2 \mid B) = 1$$

On the other hand, if they have not been bribing judges, then the probability they stay on the show is $1/3$, independent of what happens on earlier episodes. By conditional independence, we have:

$$Pr(S_1 \cap S_2 \mid \overline{B}) = Pr(S_1 \mid \overline{B}) Pr(S_2 \mid \overline{B}) = \frac{1}{3} \cdot \frac{1}{3}$$

Plugging our results above into equation (1) gives us:

$$\text{Pr}(S_1 \cap S_2) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \boxed{\frac{1}{3}}$$

(c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

Solution:

By the definition of conditional probability and the Law of Total Probability,

$$\text{Pr}(\overline{S_2} \mid S_1) = \frac{\text{Pr}(S_1 \cap \overline{S_2})}{\text{Pr}(S_1)} = \frac{\text{Pr}(S_1 \cap \overline{S_2} \mid B)\, \text{Pr}(B) + \text{Pr}(S_1 \cap \overline{S_2} \mid \overline{B})\, \text{Pr}(\overline{B})}{\text{Pr}(S_1)}$$

We have already computed $P(S_1)$ in part (a). We compute the numerator term by term. Given that a contestant is bribing the judges, they are guaranteed to stay on the show. As such:

$$\text{Pr}(S_1 \cap \overline{S_2} \mid B) = \text{Pr}(S_1 \mid B) \cdot \text{Pr}(\overline{S_2} \mid B) = 1 \cdot 0 = 0$$

On the other hand, if they have not been bribing judges, the probability they leave the show is $2/3$ (by complementing). We can then write (by conditional independence on the event that they do not bribe):

$$\text{Pr}(S_1 \cap \overline{S_2} \mid \overline{B}) = \text{Pr}(S_1 \mid \overline{B}) \cdot \text{Pr}(\overline{S_2} \mid \overline{B}) = \frac{1}{3} \cdot \frac{2}{3}$$

We can now evaluate our initial expression:

$$\text{Pr}(\overline{S_2} \mid S_1) = \frac{0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4}}{\frac{1}{2}} = \frac{1/6}{1/2} = \boxed{\frac{1}{3}}$$

(d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges? **Solution:**

Let B be the event that she bribed the judges. By Bayes' Theorem,

$$\Pr(B \mid S_1) = \frac{\Pr(S_1 \mid B)\Pr(B)}{\Pr(S_1)} = \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} = \boxed{\frac{1}{2}}$$

5. Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

(a) What is the probability the system is functioning? **Solution:**

Let C_i be the event component i is working, and F be the event that the system is functioning.

For the system to function, it is sufficient for any component to be working. This means that the only case in which the system does not function is when none of the components work. We can then use complementing to compute $\Pr(F)$, knowing that $\Pr(C_i) = p$. We get:

$$\Pr(F) = 1 - \Pr(F^C) = 1 - \Pr(\bigcap_{i=1}^{n} C_i^C) = 1 - \prod_{i=1}^{n} \Pr(C_i^C) =$$

$$1 - \prod_{i=1}^{n} (1 - \Pr(C_i)) = 1 - \prod_{i=1}^{n} (1-p) = \boxed{1-(1-p)^n}$$

Note that $\Pr(\bigcap_{i=1}^{n} C_i^C) = \prod_{i=1}^{n} \Pr(C_i^C)$ due to independence of C_i (components working independently of each other). Note also that $\prod_{i=1}^{n} a = a^n$ for any constant a.

(b) If the system is functioning, what is the probability that component 1 is working? **Solution:**

We know that for the system to function only one component needs to be working, so for all i, we have $\Pr(F \mid C_i) = 1$. Using Bayes Theorem, we get:

$$\Pr(C_1|F) = \frac{\Pr(F|C_1)\Pr(C_1)}{\Pr(F)} = \frac{1 \cdot p}{1-(1-p)^n} = \boxed{\frac{p}{1-(1-p)^n}}$$

(c) If the system is functioning and component 2 is working, what is the probability that component 1 is working? **Solution:**

$$\Pr(C_1|C_2, F) = \Pr(C_1|C_2) = \Pr(C_1) = p$$

where the first equality holds because knowing C_2 and F is just as good as knowing C_2 (since if C_2 happens, F does too), and the second equality holds because the components working are independent of each other.

More formally, we can use the definition of conditional probability along with a careful application of the chain rule to get the same result. We start with the following expression:

$$\Pr(C_1 \mid C_2, F) = \frac{\Pr(C_1, C_2, F)}{\Pr(C_2, F)} = \frac{\Pr(F \mid C_1, C_2) \cdot P(C_1 \mid C_2) \Pr(C_2)}{\Pr(F \mid C_2) \cdot \Pr(C_2)}$$

We note that the system is guaranteed to work if any one component is working, so $\Pr(F \mid C_1, C_2) = \Pr(F|C_2) = 1$. We also note that components work independently of each other, hence $\Pr(C_1|C_2) = \Pr(C_1)$. With that in mind, we can rewrite our expression so that:

$$\Pr(C_1 \mid C_2, F) = \frac{1 \cdot \Pr(C_1) \cdot \Pr(C_2)}{1 \cdot \Pr(C_2)} = \Pr(C_1) = \boxed{p}$$

6. Allergy Season

In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy? **Solution:**

Let E be the event that the drug is effective for Sneezy, and C_i be the event that he gets i colds the first winter. By Bayes' Theorem,

$$\Pr(E \mid C_1) = \frac{\Pr(C_1 \mid E)\Pr(E)}{\Pr(C_1 \mid E)\Pr(E) + \Pr(C_1 \mid \overline{E})\Pr(\overline{E})} = \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.2 \times 0.8} = \frac{3}{11}$$

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy? **Solution:**

Let the reduced sample space for part (b) be C_1 from part (a), so that $\Pr_{C_1}(E) = \Pr_{\Omega}(E|C_1)$. Let D_i be the event that he gets i colds the second winter. By Bayes' Theorem,

$$\Pr(E \mid D_2) = \frac{\Pr(D_2 \mid E)\Pr(E)}{\Pr(D_2 \mid E)\Pr(E) + \Pr(D_2 \mid \overline{E})\Pr(\overline{E})} = \frac{0.2 \times \frac{3}{11}}{0.2 \times \frac{3}{11} + 0.2 \times \frac{8}{11}} = \frac{3}{11}$$

(c) Why is the answer to (b) the same as the answer to (a)? **Solution:**

The probability of two colds whether or not the drug was effective is the same. Hence knowing that Sneezy got two colds does not change the probability of the drug's effectiveness.

7. A game

Howard and Jerome are playing the following game: A 6-sided die is thrown and each time it's thrown, regardless of the history, it is equally likely to show any of the six numbers.

- If it shows 5, Howard wins.
- If it shows 1, 2, or 6, Jerome wins.
- Otherwise, they play a second round and so on.

What is the probability that Jerome wins on the 4th round? **Solution:**

Let S_i be the event that Jerome wins on the i-th round and let N_i be the event that nobody wins on the i-th round. Then we are interested in the event

$$N_1 \cap N_2 \cap N_3 \cap S_4.$$

Using the chain rule, we have

$$\begin{aligned}\Pr(N_1, N_2, N_3, S_4) &= \Pr(N_1) \cdot \Pr(N_2|N_1) \cdot \Pr(N_3|N_1, N_2) \cdot \Pr(S_4|N_1, N_2, N_3) \\ &= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2}.\end{aligned}$$

In the final step, we used the fact that if the game hasn't ended, then the probability that it continues for another round is the probability that the die comes up 3 or 4, which has probability $1/3$.

8. Another game

Alice and Alicia are playing a tournament in which they stop as soon as one of them wins n games. Alicia wins each game with probability p and Alice wins with probability $1-p$, independently of other games. What is the probability that Alicia wins and that when the match is over, Alice has won k games?

Solution:

Since the match is over when someone wins the n^{th} game, and Alicia won the match, Alicia won the last game. Before this, Alicia must've won $n-1$ games and Alice must've won k games. Therefore, the probability that we reach a point in time when Alicia has won $n-1$ games and Alice has won k games is: $p^{n-1} \cdot (1-p)^k \cdot \binom{n-1+k}{k}$. The binomial coefficient counts the number of ways of picking the k games that Alice has won out of $n-1+k$ games.
At that point in time, we want Alicia to win the next game so that she has won n games. This happens with probability p, independent of previous outcomes. Therefore, our final probability is:

$$p^{n-1} \cdot (1-p)^k \cdot \binom{n-1+k}{k} \cdot p = p^n \cdot (1-p)^k \cdot \binom{n-1+k}{k}$$

9. Dependent Dice Duo

This problem demonstrates that independence can be "broken" by conditioning. Let D_1 and D_2 be the outcomes of two independent rolls of a fair die. Let E be the event "$D_1 = 1$", F be the event "$D_2 = 6$", and G be the event "$D_1 + D_2 = 7$". Even though E and F are independent, show that

$$\mathbb{P}(E \cap F \mid G) \neq \mathbb{P}(E \mid G)\,\mathbb{P}(F \mid G).$$

Solution:

When we condition on G our sample space Ω becomes $\{(1,6),(2,5),(3,4),(4,3),(2,5),(6,1)\}$ where the first number in the pair is the D_1 outcome and the second number is the D_2 outcome.

From here we can see that $\mathbb{P}(E \mid G) = \mathbb{P}(D_1 = 1 \mid D_1 + D_2 = 7) = 1/6$ as $(1,6)$ is 1 of the 6 possible rolls that sum to 7 and each roll is equally likely. We also see that $\mathbb{P}(E \mid G) = \mathbb{P}(D_1 = 1 \mid D_1 + D_2 = 7) = 1/6$ and $\mathbb{P}(E \cap F \mid G) = \mathbb{P}(D_1 = 1 \cap D_2 = 6 \mid D_1 + D_2 = 7) = 1/6$ using similar reasoning.

Now we have that $\mathbb{P}(E \mid G) * \mathbb{P}(F \mid G) = 1/36$.

Notice that $\frac{1}{36} \neq \frac{1}{6}$, so we have shown that independence can be "broken" by conditioning.

10. Infinite Lottery

Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\Pr(A_k) = (\frac{1}{2})^k$. Once we generate a number k, that is the maximum we can win. That is, after generating a value k, we can win any number in $[k] = \{1, ..., k\}$ dollars. Suppose the probability that we win $\$j$ for $j \in [k]$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly $\$1$. Given that we win exactly one dollar, what is the probability that the number generated was also 1? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$for $a > 1$.

Solution:

The probability that we are looking for is $\Pr(A_1|B)$. By Bayes' Theorem and the law of total probability we see that $\Pr(A_1|B) = \frac{\Pr(B|A_1)\Pr(A_1)}{\sum_{j=1}^{\infty}\Pr(B|A_j)\Pr(A_j)}$. Using the formulas given in the problem statement we get:

$$\Pr(A_1|B) = \frac{\Pr(B|A_1)\Pr(A_1)}{\sum_{j=1}^{\infty}\Pr(B|A_j)\Pr(A_j)} = \frac{\frac{1}{1}\frac{1}{2^1}}{\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{2^j}} = \frac{1}{2\ln 2} \approx \boxed{0.7213}$$

11. The Monty Hall Problem

The Monty Hall problem is a famous, seemingly counter-intuitive probability puzzle named after Monty Hall, the host of the show "Let's Make a Deal". This problem emphasizes the importance of using given information to make decisions.

Assume you are a contestant on this game show. In the original problem, there are 3 doors, one hiding a car and the other two hiding goats. At first, you randomly pick a door, hoping you can win the car. As Monty knows exactly what door hides the location of the car, he purposefully decides to reveal a door different from your pick which is guaranteed to reveal a goat. As there are 2 doors left, Monty later asks if you want to stick to your current door or to switch to the other door.

In the beginning, when there is no information about these 3 doors, every door has equal probability of revealing a car. However, after knowing that Monty will only open a door which definitely reveals a goat, it turns out that switching to the other door yields a higher probability of winning than sticking to your current door. Thus, the best strategy is to switch to the other door. Feel free to do any calculations on your own to find out why.

For this problem, you have to determine the best strategy when there are 4 doors. As a contestant, you first randomly choose a door. Monty opens one of the 3 other doors, which reveals a goat, and asks if you want to stick to your current choice or switch to a different door. After you make your pick, Monty opens another door (other than your current pick) which also reveals a goat. This time, you have to make the final pick: sticking to the current door in the previous pick or switching to the other door. Make a thorough analysis of all possible strategies and explain which one is the best.

Solution:

We calculate probability of winning given that we play with a certain strategy. We use R and W to indicate when you pick the right and wrong door at a specific pick, respectively. For example, $P_1 = R, P_2 = R, P_3 = R$ means that you choose the right door in all 3 picks.

Note that in this solution, we use the semicolon notation:

$$\mathbb{P}(P_1 = R, P_2 = R, P_3 = R; S_1)$$

to indicate the probability of 3 right picks under strategy S_1, instead of:

$$\mathbb{P}(P_1 = R, P_2 = R, P_3 = R|S_1)$$

i.e. "probability of 3 right picks conditioned on strategy S_1", because a strategy is not a random variable.

For each strategy S_i, we want to calculate the probability of winning a car, which means when the third pick is right, i.e. $\mathbb{P}(P_3 = R; S_i)$.

(a) S_1: Stick-and-stick strategy. There are only 2 cases, RRR and WWW. We only need to calculate the case RRR.

For $P(P_1 = R, P_2 = R, P_3 = R; S_1)$:

- $P(P_1 = R; S_1) = \frac{1}{4}$, because the probability of being correct in a pick is $\frac{1}{4}$
- $P(P_2 = R|P_1 = R; S_1) = 1$, because you have to stick to your first pick.
- $P(P_3 = R|P_2 = R, P_1 = R; S_1) = 1$, because you have to stick to your second pick.

Thus:

$$\begin{aligned}\mathbb{P}(\text{win}; S_1) &= \mathbb{P}(P_1 = R, P_2 = R, P_3 = R; S_1) \\ &= \mathbb{P}(P_1 = R; S_1)\mathbb{P}(P_2 = R|P_1 = R; S_1)\mathbb{P}(P_3 = R|P_1 = R, P_2 = R; S_1) \\ &= \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4}\end{aligned}$$

(b) S_2: Stick-and-switch strategy. There are only 2 cases, WWR and RRW. Thus, we only need to calculate the probability for the case WWR.

For $P(P_1 = W, P_2 = W, P_3 = R; S_2)$:

- $P(P_1 = W; S_2) = \frac{3}{4}$, because the probability of being wrong in a pick is $\frac{3}{4}$
- $P(P_2 = W|P_1 = W; S_2) = 1$, because you have to stick to your first pick.
- $P(P_3 = R|P_2 = W, P_1 = W; S_2) = 1$, because conditioned on two previous wrong doors, there is only one right door left out of 2. Monty will show the wrong door in his second reveal anyway, so you're guaranteed to pick the right door if you switch.

Thus:

$$\begin{aligned}\mathbb{P}(\text{win}; S_2) &= \mathbb{P}(P_1 = W, P_2 = W, P_3 = R; S_2) \\ &= \mathbb{P}(P_1 = W; S_2)\mathbb{P}(P_2 = W|P_1 = W; S_2)\mathbb{P}(P_3 = R|P_1 = W, P_2 = W; S_2) \\ &= \frac{3}{4} \cdot 1 \cdot 1 = \frac{3}{4}\end{aligned}$$

(c) S_3: Switch-and-stick strategy

There are 3 cases, RWW, WWW and WRR. However, we only need to calculate the probability for WRR.

For $P(P_1 = W, P_2 = R, P_3 = R; S_3)$:

- $P(P_1 = W; S_3) = \frac{3}{4}$, because the probability of being wrong in a pick is $\frac{3}{4}$
- $P(P_2 = R|P_1 = W; S_3) = \frac{1}{2}$, because conditioned on the first wrong door, and during first reveal Monty will show a wrong door, there are two remaining doors to switch to, one of which will be correct.
- $P(P_3 = R|P_1 = W, P_2 = R; S_3) = 1$, because conditioned on the second pick, which is correct, if you stick to it, you're guaranteed to pick the right door.

Thus:

$$\begin{aligned}\mathbb{P}(\text{win}; S_3) &= \mathbb{P}(P_1 = W, P_2 = R, P_3 = R; S_3) \\ &= \mathbb{P}(P_1 = W; S_3)\mathbb{P}(P_2 = R|P_1 = W; S_3)\mathbb{P}(P_3 = R|P_1 = W, P_2 = R; S_3) \\ &= \frac{3}{4} \cdot \frac{1}{2} \cdot 1 = \frac{1}{4} = \frac{3}{8}\end{aligned}$$

(d) S4: Switch-and-switch strategy

There are 3 cases, RWR, WWR and WRW. However, we only need to calculate the probabilities for RWR and WWR.

For $P(P_1 = R, P_2 = W, P_3 = R; S_4)$:

- $P(P_1 = R; S_4) = \frac{1}{4}$, because the probability of being right in a pick is $\frac{1}{4}$
- $P(P_2 = W|P_1 = R; S_4) = 1$, because conditioned on the first right door, if you switch you're guaranteed to pick a wrong door.
- $P(P_3 = R|P_1 = R, P_2 = W; S_4) = 1$, because conditioned on the second wrong pick and two wrong doors have been opened by Monty, if you switch you're guaranteed to pick the right one.

$$\begin{aligned}&\mathbb{P}(P_1 = R, P_2 = W, P_3 = R; S_4) \\ &\qquad = \mathbb{P}(P_1 = R; S_4)\mathbb{P}(P_2 = W|P_1 = R; S_4)\mathbb{P}(P_3 = R|P_1 = R, P_2 = W; S_4) \\ &\qquad = \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4} = \frac{1}{4}\end{aligned}$$

For $P(P_1 = W, P_2 = W, P_3 = R; S_4)$:

- $P(P_1 = W; S_4) = \frac{3}{4}$, because the probability of being wrong in a pick is $\frac{3}{4}$
- $P(P_2 = W|P_1 = W; S_4) = \frac{1}{2}$, because conditioned on the first wrong pick and Monty has opened a wrong door, there is a right door to switch to out of the 2 remaining doors.
- $P(P_3 = R|P_1 = W, P_2 = W; S_4) = 1$, because conditioned on the second wrong pick and 2 wrong doors have been opened by Monty, if you switch to the remaining door it is guaranteed to be right.

$$\begin{aligned}&\mathbb{P}(P_1 = W, P_2 = W, P_3 = R; S_4) \\ &\qquad = \mathbb{P}(P_1 = W; S_4)\mathbb{P}(P_2 = W|P_1 = W; S_4)\mathbb{P}(P_3 = R|P_1 = W, P_2 = W; S_4) \\ &\qquad = \frac{3}{4} \cdot \frac{1}{2} \cdot 1 = \frac{3}{8}\end{aligned}$$

Thus:

$$\begin{aligned}\mathbb{P}(\text{win}; S_4) &= \mathbb{P}(P_1 = R, P_2 = W, P_3 = R; S_4) + \mathbb{P}(P_1 = W, P_2 = W, P_3 = R; S_4) \\ &= \frac{1}{4} + \frac{3}{8} = \frac{5}{8}\end{aligned}$$

In conclusion, stick-and-switch strategy is the best strategy.

Solution:

The optimal strategy is to switch doors only on the very last move.
When you make your first choice (out of 4 doors), you have a $\frac{1}{4}$ chance of selecting the correct door. This probability holds up throughout the entire game, even as more doors with goats are opened, because at the moment you selected it, you only had a $\frac{1}{4}$ chance of success. So if you stick with this door throughout the game, you have a $\frac{1}{4}$ chance of winning.
When you choose your first door, there is a $\frac{3}{4}$ chance one of the other 3 doors holds the car. So when the host eliminates one of these doors by revealing the first goat, there is now a $\frac{3}{4}$ chance of the car being behind one of 2 doors. Each of these 2 doors has an equal probability of holding the car, so a probability of $\frac{3}{8}$ each.
Now comes the interesting part. When the first goat is revealed, we are given the opportunity switch doors. If we switch doors, we will have a $\frac{3}{8}$ chance of selecting the correct one, which is higher than $\frac{1}{4}$. So should we switch? Not so fast. If we switch, it means the other two doors have a combined $\frac{5}{8}$ chance of holding the car (since we selected the winning door with probability $\frac{3}{8}$). The host will then reveal a second goat, leaving us with 2 choices of doors. Our current door wins with probability $\frac{3}{8}$, and the other door wins with probability $\frac{5}{8}$. So the best we can do is win with probability $\frac{5}{8}$.
But what if we never switched doors after the first goat was revealed? In this case, our current door only has a $\frac{1}{4}$ chance of winning, and when the host reveals a second goat, the other remaining door has a $\frac{3}{4}$ chance of holding the car! This represents a better chance of winning than any previous strategy.
In conclusion, we should wait to switch until the very last phase, and then switch to win with probability $\frac{3}{4}$.

12. Flipping Coins

A coin is tossed twice. The coin is "heads" one quarter of the time. You can assume that the second toss is independent of the first toss.

(a) What is the probability that the second toss is "heads" given that the first toss is "tails"? **Solution:**

Consider the probability space with sample space $\Omega = \{HH, TT, HT, TH\}$. Because heads come $1/4$ of the time, and tails $3/4$, we have $\mathbb{P}(HH) = 1/4 \times 1/4 = 1/16$, $\mathbb{P}(HT) = \mathbb{P}(TH) = 3/4 \times 1/4 = 3/16$ and finally $\mathbb{P}(TT) = 9/16$.

Then, let A be the event that the first coin is tails, and let B be the event that the second coin is heads. Then,

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} .$$

Note that $A = \{TT, TH\}$ and $B = \{HH, TH\}$, and thus

$$\mathbb{P}(A) = \mathbb{P}(TT) + \mathbb{P}(TH) = 9/16 + 3/16 = 12/16 = 3/4$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(TH) = 3/16 .$$

Therefore, $\mathbb{P}(B|A) = (3/16)/(3/4) = 1/4$.

It is important to realize that this exactly what we would have expected – indeed, we model the coins to be independent.

(b) What is the probability that the second toss is "heads" given that at least one of the tosses is "tails"? **Solution:**

Let A, B be the same events as in **a)**. We define $C = \{TH, TT, TT\}$, and we want $\mathbb{P}(B|C)$. Note that

$$\mathbb{P}(C) = 1 - \mathbb{P}(HH) = 15/16$$
$$\mathbb{P}(B \cap C) = \mathbb{P}(TH) = 3/16 \ .$$

Therefore,

$$\mathbb{P}(B \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{3/16}{15/16} = \frac{3}{15} = \frac{1}{5} \ .$$

(c) In the probability space of this task, give an example of two events that are disjoint but not independent.
Solution:

$E_1 = \{TT\}$ and $E_2 = \{HH\}$ are disjoint, but not independent. Indeed, $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(\emptyset) = 0$, but each event occurs with positive probability, and so $\mathbb{P}(E_1) \cdot \mathbb{P}(E_2) > 0$.

(d) In the probability space of this task, give an example of two events that are independent but not disjoint.
Solution:

$E_1 = \{TH, HH\}$ and $E_2 = \{TH, TT\}$ are not disjoint, but are independent.

13. Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

(a) What is the probability that all three balls are all of the same color? **Solution:**

The experiment is modeled with $\Omega = \{r, b\}^3$. Probabilities are assigned as we have seen in class, by assuming every draw is uniform among the remaining balls. Then, note that $\mathbb{P}(rrr) = 3/7 \cdot 2/6 \cdot 1/5 = 1/35$ and $\mathbb{P}(bbb) = 4/7 \cdot 3/6 \cdot 2/5 = 4/35$. Therefore, the probability that they all have the same color is $1/35 + 4/35 = 1/7$.

(b) What is the probability that we get more than one red ball given the first ball is red?

Solution:

Let R be the event that the first ball is red. Since we select the first ball uniformly, $\mathbb{P}(R) = \frac{3}{7}$. (This can be computed explicitly from Ω.) We also consider the event M that we have more than one red ball. Let M be the event that more than one ball is red. We need to now compute the probability $\mathbb{P}(M \cap R)$, but note that by the law of total probability

$$\mathbb{P}(M \cap R) = \mathbb{P}(R) - \mathbb{P}(M^c \cap R) = 3/7 - \mathbb{P}(M^c \cap R) \ .$$

We could compute this probability directly from Ω, but there is an easier way. Note that $M^c \cap R$ is the event that the first ball is red, and both remaining balls are blue. In particular,

$$\mathbb{P}(M^c \cap R) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{6}{35} \ .$$

Thus, $\mathbb{P}(M \cap R) = 3/7 - 6/35 = 9/35$, and

$$\mathbb{P}(M|R) = \frac{\mathbb{P}(M \cap R)}{\mathbb{P}(R)} = \frac{9/35}{3/7} = \frac{3}{5}\ .$$