- PDF/CDF review
- $\mathbb{E}$ details
- more examples of cont r.v.s.

Continuous Zoo

CSE 312 Spring 24
Lecture 16

Let's start with the pmf

For discrete random variables, we defined the pmf: $p_Y(k) = \mathbb{P}(Y = k)$.

We can't have a pmf quite like we did for discrete random variables. Let X be a random real number between 0 and 1.

$\mathbb{P}(X = .1) = \frac{1}{\infty}$??

Let's try to maintain as many rules as we can…

Discrete	Continuous
$p_Y(k) \geq 0$	$f_X(k) \geq 0$
$\sum_{\omega} p_Y(\omega) = 1$	$\int_{-\infty}^{\infty} f_X(k)\ \mathrm{d}k = 1$

Use f_X instead of p_X to remember it's different .

The probability density function

For Continuous random variables, the analogous object is the "probability density function" we write $f_X(k)$ instead of $p_X(k)$

Idea: Make it "work right" for **events** since single outcomes don't make sense.

$$\mathbb{P}(a \leq X \leq b) = c \qquad \boxed{\int_a^b f_X(z)\,\mathrm{d}z = c} \qquad \text{integrating is analogous to sum.}$$

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$\mathbb{P}(a \leq X \leq b) = c$ $\boxed{\int_a^b f_X(z)\,\mathrm{d}z = c}$ integrating is analogous to sum.

PDF for uniform

Let X be a uniform real number between $\mathbf{0}$ and $\mathbf{1}$.

What should $f_X(k)$ be to make all those events integrate to the right values?

$$f_X(k) = \begin{cases} 0 & \text{if } k < 0 \text{ or } k > 1 \\ 1 & \text{if } 0 \le k \le 1 \end{cases}$$

Probability Density Function

So $\mathbb{P}(X = .1) = ??$

$f_X(.1) = 1$

$$\int_{.1}^{.1} f_X(z)\,dz = 0$$

The number that best represents $\mathbb{P}(X = .1)$ is 0.

This is different from $f_X(x)$

For continuous probability spaces:
Impossible events have probability 0,
but some probability 0 events might be possible.

So…what is $f_X(x)$???

Using the PDF

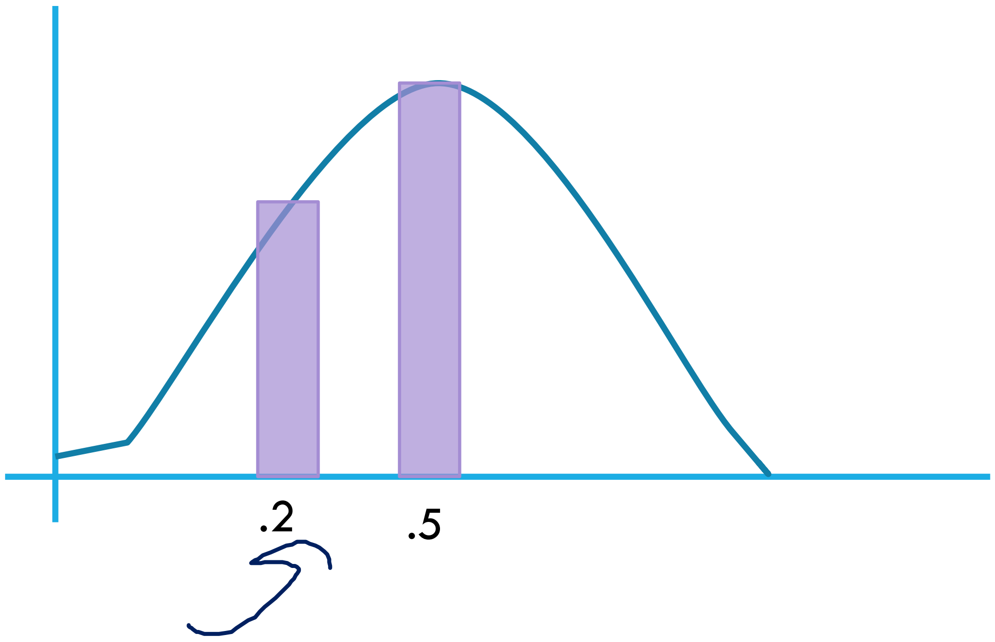
Let's look at a different pdf…

Compare the events: $X \approx .2$ and $X \approx .5$

$\mathbb{P}(.2 - \epsilon/2 \leq X \leq .2 + \epsilon/2)$

What will the pdf give? $\int_{.2-\epsilon/2}^{-2+\epsilon/2} f_X(z)\,\mathrm{d}z$

$f_X(.2) \cdot \epsilon$



What happens if we look at the ratio

$\frac{\mathbb{P}(X\approx.2)}{\mathbb{P}(X\approx.5)}$

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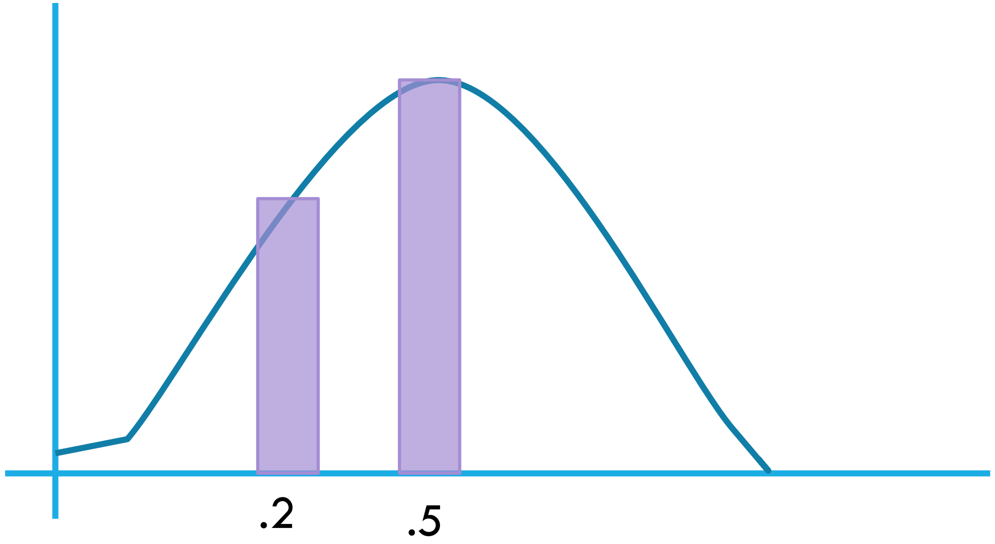

What happens if we look at the ratio

$$\frac{\mathbb{P}(X \approx .2)}{\mathbb{P}(X \approx .5)} = \frac{\mathbb{P}\left(.2 - \frac{\epsilon}{2} \leq X \leq .2 + \frac{\epsilon}{2}\right)}{\mathbb{P}\left(.5 - \frac{\epsilon}{2} \leq X \leq .5 + \frac{\epsilon}{2}\right)} = \frac{\epsilon f_X(.2)}{\epsilon f_X(.5)} = \frac{f_X(.2)}{f_X(.5)}$$

So what’s the pdf?

It’s the number that when integrated over gives the probability of an event.

Equivalently, it’s number such that:

-integrating over all real numbers gives 1.

-comparing $f_X(k)$ and $f_X(\ell)$ gives the relative chances of X being near k or ℓ.

CDFs

What's a CDF?

The Cumulative Distribution Function $F_X(k) = \mathbb{P}(\boldsymbol{X} \leq \boldsymbol{k})$ analogous to the CDF for discrete variables.

$$F_X(k) = \mathbb{P}(X \leq k) = \int_{-\infty}^{k} f_X(z)\,\mathrm{d}z$$

So how do I get from CDF to PDF? Taking the derivative!

$$\frac{\mathrm{d}}{\mathrm{d}k} F_X(k) = \frac{\mathrm{d}}{dk}\left(\int_{-\infty}^{k} f_X(z)\,\mathrm{d}z\right) = f_X(k)$$

Comparing Discrete and Continuous

	Discrete Random Variables	Continuous Random Variables
Probability 0	Equivalent to impossible	All impossible events have probability 0, but not conversely.
Relative Chances	PMF: $p_X(k) = \mathbb{P}(X = k)$	PDF $f_X(k)$ gives chances relative to $f_X(k')$
Events	Sum over PMF to get probability	Integrate PDF to get probability
Convert from CDF to PMF	Sum up PMF to get CDF. Look for "breakpoints" in CDF to get PMF.	Integrate PDF to get CDF. Differentiate CDF to get PDF.
$\mathbb{E}[X]$	$\sum_{\omega} X(\omega) \cdot p_X(\omega)$	$\int_{-\infty}^{\infty} z \cdot f_X(z)\, \mathrm{d}z$
$\mathbb{E}[g(X)]$	$\sum_{\omega} g(X(\omega)) \cdot p_X(\omega)$	$\int_{-\infty}^{\infty} g(z) \cdot f_X(z)\, \mathrm{d}z$
$\mathrm{Var}(X)$	$\mathbb{E}[X^2] - (\mathbb{E}[X])^2$	$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{-\infty}^{\infty} (z - \mathbb{E}[X])^2 f_X(z)\, \mathrm{d}z$

What about expectation?

For a random variable X, we define:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} X(z) \cdot f_X(z)\, \mathrm{d}z$$

Just replace summing over the pmf with integrating the pdf.

It still represents the average value of X.

Expectation of a function

LOTUS

For any function g and any continuous random variable, X:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(X(z)) \cdot f_X(z)\, \mathrm{d}z$$

Again, analogous to the discrete case; just replace summation with integration and pmf with the pdf.

We're going to treat this as a definition.

Technically, this is really a theorem; since $f()$ is the pdf of X and it only gives relative likelihoods for X, we need a proof to guarantee it "works" for $g(X)$.

Sometimes called "Law of the Unconscious Statistician."

Linearity of Expectation

$\mathbb{E}[X+Y]$

Still true!

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

For all X, Y; even if they're continuous.

Won't show you the proof – for just $\mathbb{E}[aX + b]$, it's

$$\mathbb{E}[aX + b] = \int_{-\infty}^{\infty}[aX(k) + b]f_X(k)\,\mathrm{d}k$$

$$= \int_{-\infty}^{\infty} aX(k)f_X(k)dk + \int_{-\infty}^{\infty} bf_X(k)dk$$

$$= a\int_{-\infty}^{\infty} X(k)f_X(k)dk + b\int_{-\infty}^{\infty} f_X(k)dk$$

$$= a\mathbb{E}[X] + b$$

Variance

$$\mathbb{E}\left[\left(X-\mathbb{E}[X]\right)^2\right]$$

No surprises here

$$\mathbf{Var}(\boldsymbol{X}) = \mathbb{E}[\boldsymbol{X}^2] - (\mathbb{E}[\boldsymbol{X}])^2 = \int_{-\infty}^{\infty} \boldsymbol{f_X}(\boldsymbol{k})(\boldsymbol{X}(\boldsymbol{k}) - \mathbb{E}[\boldsymbol{X}])^2\, \mathbf{d}\boldsymbol{k}$$

Let's calculate an expectation

Let X be a uniform random number between a and b.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z)\, \mathrm{dz}$$

$$= \int_{-\infty}^{a} z \cdot 0\, \mathrm{dz} + \int_{a}^{b} z \cdot \frac{1}{b-a} \mathrm{dz} + \int_{b}^{\infty} z \cdot 0\, \mathrm{dz}$$

$$= 0 + \int_{a}^{b} \frac{z}{b-a}\, \mathrm{dz} + 0$$

$$= \frac{z^2}{2(b-a)}\bigg|_{z=a}^{b} = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

What about $\mathbb{E}[g(X)]$

Let $X \sim \text{Unif}(a, b)$, what about $\mathbb{E}[X^2]$?

$$f_X(z) = \begin{cases} \frac{1}{b-a} & a \le z \le b \\ 0 & \text{o/w} \end{cases}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} z^2 f_X(z) \mathrm{dz}$$

$$= \int_{-\infty}^{a} z^2 \cdot 0 \, \mathrm{dz} + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \mathrm{dz} + \int_{b}^{\infty} z^2 \cdot 0 \, \mathrm{dz}$$

$$= 0 + \int_{a}^{b} z^2 \cdot \frac{1}{b-a} \mathrm{dz} + 0$$

$$= \frac{1}{b-a} \cdot \frac{z^3}{3}\Big|_{z=a}^{b} = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{1}{3(b-a)} \cdot (b-a)(a^2 + ab + b^2)$$

$$= \frac{a^2+ab+b^2}{3}$$

Let's assemble the variance

$$\mathrm{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4(a^2+ab+b^2)}{12} - \frac{3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12}$$

$$= \frac{(a-b)^2}{12}$$

Continuous Uniform Distribution

$X \sim \text{Unif}(a, b)$ (uniform real number between a and b)

PDF: $f_X(k) = \begin{cases} \frac{1}{b-a} & \text{if } a \le k \le b \\ 0 & \text{otherwise} \end{cases}$

CDF: $F_X(k) = \begin{cases} 0 & \text{if } k < a \\ \frac{k-a}{b-a} & \text{if } a \le k \le b \\ 1 & \text{if } k \ge b \end{cases}$

$\mathbb{E}[X] = \frac{a+b}{2}$

$\text{Var}(X) = \frac{(b-a)^2}{12}$

Continuous Zoo

$X \sim \text{Unif}(a, b)$	$X \sim \text{Exp}(\lambda)$	$X \sim \mathcal{N}(\mu, \sigma^2)$
$f_X(k) = \frac{1}{b-a}$ $\mathbb{E}[X] = \frac{a+b}{2}$ $\text{Var}(X) = \frac{(b-a)^2}{12}$	$f_X(k) = \lambda e^{-\lambda k}$ for $k \geq 0$ $\mathbb{E}[X] = \frac{1}{\lambda}$ $\text{Var}(X) = \frac{1}{\lambda^2}$	$f_X(k) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $\mathbb{E}[X] = \mu$ $\text{Var}(X) = \sigma^2$

It's a smaller zoo, but it's just as much fun!

Exponential Random Variable

Like a geometric random variable, but continuous time. How long do we wait until an event happens? (instead of "how many flips until a heads")

Where waiting doesn't make the event happen any sooner.

Geometric: $\mathbb{P}(X = k + 1 \,|X \geq 1) = \mathbb{P}(X = k)$

When the first flip is tails, the coin doesn't remember it came up tails, you've made no progress.

For an exponential random variable:

$\mathbb{P}(X \geq k + 1 \,|X \geq 1) = \mathbb{P}(Y \geq k)$

Exponential random variable

If you take a Poisson random variable and ask "what's the time until the next event" you get an exponential distribution!

Let's find the CDF for an exponential.

Let $Y \sim \text{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit.

What's $\mathbb{P}(Y > t)$?

What Poisson are we waiting on, and what event for it tells you that $Y > t$?

Exponential random variable

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Let's find the CDF for an exponential.

Let $Y \sim \text{Exp}(\lambda)$, be the time until the first event, when we see an average of λ events per time unit. What's $\mathbb{P}(Y > t)$?

What Poisson are we waiting on? For $X \sim \text{Poi}(\lambda t)$ $\mathbb{P}(Y > t) = \mathbb{P}(X = 0)$

$$\mathbb{P}(X = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

$F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$ (for $t \geq 0$, $F_Y(x) = 0$ for $x < 0$)

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

What's the density?

$f_Y(t) =$

Find the density

We know the CDF, $F_Y(t) = \mathbb{P}(Y \leq t) = 1 - e^{-\lambda t}$

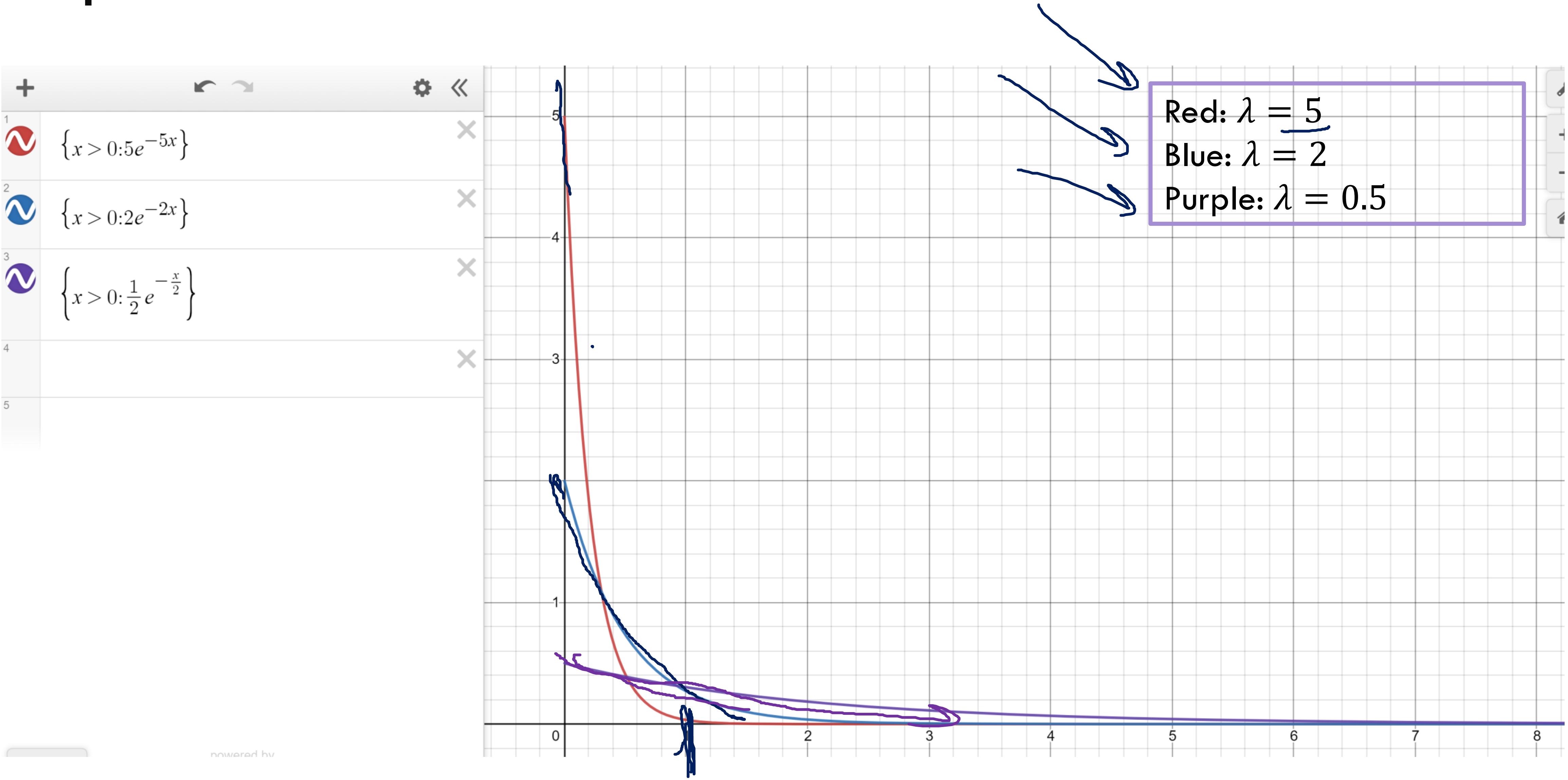
What's the density?

$$f_Y(t) = \frac{d}{dt}\left(1 - e^{-\lambda t}\right) = 0 - \frac{d}{dt}\left(e^{-\lambda t}\right) = \lambda e^{-\lambda t}.$$

For t ≥ 0 it's that expression

For $t < 0$ it's just 0.

Exponential PDF



Memorylessness

$$\int_{k+1}^{\infty} f_X(z)\,dz$$

$$\mathbb{P}(X \geq k+1 | X \geq 1) = \frac{\mathbb{P}(X\geq k+1 \cap X \geq 1)}{\mathbb{P}(X\geq 1)} = \frac{\mathbb{P}(X\geq k+1)}{1-(1-e^{-\lambda\cdot 1})}$$

$$1 - F_X(1)$$

$$e^{-\lambda 1}$$

$$= \frac{e^{-\lambda(k+1)}}{e^{-\lambda}} = e^{-\lambda k}$$

$$\mathbb{P}(X \leq 1) - \cancel{\mathbb{P}(X=1)} = \mathbb{P}(X<1)$$

What about $\mathbb{P}(X \geq k)$ (without conditioning on the first step)?

$1-(1-e^{-\lambda k}) = e^{-\lambda k}$

It's the same!!!

More generally, for an exponential rv X, $\mathbb{P}(X \geq s+t | X \geq s) = \mathbb{P}(X \geq t)$

Side note

I hid a trick in that algebra,

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X < 1) = 1 - \mathbb{P}(X \leq 1)$$

The first step is the complementary law.

The second step is using that $\int_1^1 f_X(z)\mathrm{d}z = 0$

In general, for continuous random variables we can switch out $\leq$ and $<$ without anything changing.

We can't make those switches for discrete random variables.

Expectation of an exponential

Don't worry about the derivation (it's here if you're interested; you're not responsible for the derivation. Just the value.

Let $X \sim \text{Exp}(\lambda)$

$\mathbb{E}[X] = \int_{-\infty}^{\infty} z \cdot f_X(z)\text{d}z$

$= \int_0^{\infty} z \cdot \lambda e^{-\lambda z}\, dz$

Let $u = z;\ dv = \lambda e^{-\lambda z} dz\ (v = -e^{-\lambda z})$

Integrate by parts: $-ze^{-\lambda z} - \int -e^{-\lambda z}\, dz = -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}$

Definite Integral: $-ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z}|_{z=0}^{\infty} = (\lim_{z\to\infty} -ze^{-\lambda z} - \frac{1}{\lambda} e^{-\lambda z})\ - (0 - \frac{1}{\lambda})$

By L'Hopital's Rule $(\lim_{z\to\infty} -\frac{z}{e^{\lambda z}} - \frac{1}{\lambda e^{\lambda z}})\ - (0 - \frac{1}{\lambda}) = \left(\lim_{z\to\infty} - \frac{1}{\lambda e^{\lambda z}}\right) + \frac{1}{\lambda} = \frac{1}{\lambda}$

Variance of an exponential

If $\mathrm{X} \sim \mathrm{Exp}(\lambda)$ then $\mathrm{Var}(X) = \frac{1}{\lambda^2}$

Similar calculus tricks will get you there.

Exponential

$X \sim \text{Exp}(\lambda)$

Parameter $\lambda \geq 0$ is the average number of events in a unit of time.

$$f_X(k) = \begin{cases} \lambda e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(k) = \begin{cases} 1 - e^{-\lambda k} & \text{if } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$