Conditional Probability

CSE 312 Spring 24
Lecture 6

But first one more example with uniform probability spaces!

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: $\{(x, y): x \text{ and } y \text{ are different cards }\}$

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability: $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50!}{52!}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards

Probability Measure: uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability: $\frac{13 \cdot P(4,2) \cdot 50 * 49 * 48 * \cdots . * 2 * 1}{52 * 51 * 50 * 49 * 48 * \ldots * 2 * 1}$

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Few notes about events and samples spaces

- If you're dealing with a situation where you may be able to use a uniform probability space, make sure to set up the sample space in a way that every outcome is equally likely.

- Try not overcomplicate the sample space – only include the information that you need in it.

- When you define an event, make sure it is a <u>subset</u> of the sample space!

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if ?

$\mathbb{P}(E) = 1$ if and only if ?

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

Conditional Probabilities

Conditioning

You roll a fair red die and a fair blue die (without letting the dice affect each other).

But they fell off the table and you can't see the results.

I can see the results – I tell you the sum of the two dice is 4.

What's the probability that the red die shows a 5, **conditioned** on knowing the sum is 4?

Conditioning

You roll a fair red die and a fair blue die (without letting the dice affect each other).

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What's the probability that the red die shows a 5, **conditioned** on knowing the sum is 4?

It's 0.

Without the conditioning it was $1/6$.

Conditioning

When I told you "the sum of the dice is 4" we restricted the sample space.

The only remaining outcomes are $\{(1,3),(2,2),(3,1)\}$ out of $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$.

Outside the (restricted) sample space, the probability is going to become 0. What about the probabilities inside?

Conditional Probability

Conditional Probability

For an event B, with $\mathbb{P}(B) > 0$,
the "Probability of A conditioned on $\boldsymbol{B}$" is

$$\mathbb{P}(\boldsymbol{A}|\boldsymbol{B}) = \frac{\mathbb{P}(\boldsymbol{A} \cap \boldsymbol{B})}{\mathbb{P}(\boldsymbol{B})}$$

Just like with the formal definition of probability, this is pretty abstract.

It does accurately reflect what happens in the real world.

If $\mathbb{P}(B) = 0$, we can't condition on it (it can't happen! There's no point in defining probabilities where we know B has not happened) – $\mathbb{P}(A|B)$ is **undefined** when $\mathbb{P}(B) = 0$.

Conditioning…

Let A be "the red die is 5"
Let B be "the sum is 4"
Let C be "the blue die is 3"

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
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$\mathbb{P}(A|B)$

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$\mathbb{P}(A|B)$

$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0$
$\mathbb{P}(B) = 3/36$
$P(A|B) = \dfrac{0}{3/36}$

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Conditioning…

Let A be "the red die is 5"
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Let C be "the blue die is 3"

$\mathbb{P}(A|C)$

$\mathbb{P}(A \cap C) = 1/36$
$\mathbb{P}(C) = 6/36$

$$P(A|C) = \frac{1/36}{6/36}$$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
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Conditioning Practice

Red die 6 conditioned on sum 7

Red die 6 conditioned on sum 9

Sum 7 conditioned on red die 6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Take a few minutes to work on this with the people around you! (also on your handout)

Conditioning Practice

A ~ Red die 6

B ~ Sum is 7

$\mathbb{P}(A|B)$

$= \mathbb{P}(A \cap B)/P(B)$

$=$

$= 1/6$

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Conditioning Practice

A ~ Red die 6

C ~ Sum is 9

$\mathbb{P}(A|C)$

$= \mathbb{P}(A \cap C)/P(C)$

$=$

$= 1/4$

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
D1=1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1.6)
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Conditioning Practice

B ~ Sum is 7

A ~ Red die is 6

$\mathbb{P}(B|A)$

$= \mathbb{P}(B \cap A)/P(A)$

$=$

$= 1/6$

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Conditioning Practice

Red die 6 conditioned on sum 7 1/6

Red die 6 conditioned on sum 9 1/4

Sum 7 conditioned on red die 6 1/6

	D2=1	D2=2	D2=3	D2=4	D2=5	D2=6
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Direction Matters

Are $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ the same?

Direction Matters

No! $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$ are different quantities.

$\mathbb{P}($"traffic on the highway" | "it's snowing"$)$ is close to 1

$\mathbb{P}($"it's snowing" | "traffic on the highway"$)$ is much smaller; there many other times when there is traffic on the highway

It's a lot like implications – order can matter a lot!

(but there are some A, B where the conditioning doesn't make a difference)

Wonka Bars

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars.

You want to get a golden ticket. You could buy a 1000-or-so of the bars until you find one, but that's expensive…you've got a better idea!

You have a test – a very precise scale you've bought.

- If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time.
- If the bar you weigh does not have a golden ticket, the scale will (falsely) alert you only 1% of the time.

If you pick up a bar and it alerts, what is the probability you have a golden ticket?

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If you pick up a bar and it alerts, what is the probability you have a golden ticket?

Which do you think is closest to the right answer?

A. 0.1%
B. 10%
C. 50%
D. 99%

Conditioning

Let A be the event the scale ALERTS you

Let B be the event your bar has a ticket.

What probabilities are each of these?

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars.

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Conditioning

Let A be the event the scale ALERTS you

Let B be the event your bar has a ticket.

What probabilities are each of these?

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars. $\mathbb{P}(B)$

If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time. $\mathbb{P}(A|B)$

If the bar you weigh does not have a golden ticket, the scale will (falsely) alert you only 1% of the time. $\mathbb{P}(A|\bar{B})$

If you pick up a bar and it alerts, what is the probability you have a golden ticket? $\mathbb{P}(B|A)$

Conditioning

Let A be the event the scale ALERTS you

Let B be the event your bar has a ticket.

What probabilities are each of these?

$\mathbb{P}(B) = 0.1\%$
$\mathbb{P}(A|B) = 99.9\%$
$\mathbb{P}(A|\bar{B}) = 1\%$
$\mathbb{P}(B|A) = ???$

Reversing the Conditioning

All of our information conditions on whether B happens or not – does your bar have a golden ticket or not?

But we're interested in the "reverse" conditioning. We know the scale alerted us – we know the test is positive – but do we have a golden ticket?

$$\mathbb{P}(B) = 0.1\%$$
$$\mathbb{P}(A|B) = 99.9\%$$
$$\mathbb{P}(A|\bar{B}) = 1\%$$
$$\mathbb{P}(B|A) = ???$$

Bayes Rule

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Bayes Rule

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

What do we know about Wonka Bars?

$$0.999 = \frac{\mathbb{P}(B|A) \cdot \mathbb{P}(A)}{.001}$$

Filling In

What's $\mathbb{P}(A)$?

We'll use a trick called "the law of total probability":

$$\mathbb{P}(A) = \mathbb{P}(A|B) \cdot \mathbb{P}(B) + \mathbb{P}(A|\bar{B}) \cdot P(\bar{B})$$

$$= 0.999 \cdot .001 + .01 \cdot .999$$

$$= .010989$$

Law of Total Probability

Let $A_1, A_2, \ldots, A_k$ be a **partition** of Ω.

A partition of a set S is a family of subsets $S_1, S_2, \ldots, S_k$ such that:

$S_i \cap S_j = \emptyset$ for all i, j and

$S_1 \cup S_2 \cup \cdots \cup S_k = S$.

i.e. every element of Ω is in exactly one of the A_i.

Law of Total Probability

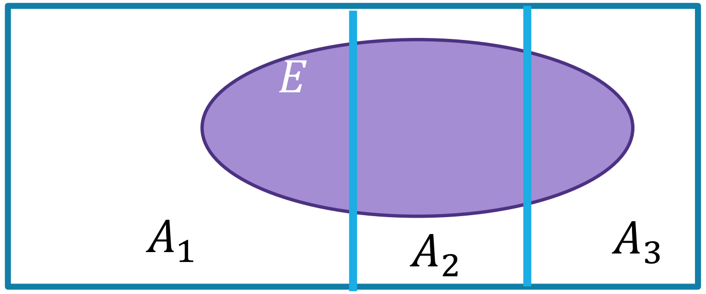
Law of Total Probability

Let $A_1, A_2, \dots, A_k$ be a **partition** of Ω.

For any event E,

$$\mathbb{P}(E) = \sum_{\text{all } i} \mathbb{P}(E|A_i)\mathbb{P}(A_i)$$

Why?



The Proof is actually pretty informative on what's going on.

$\sum_{\text{all } i} \mathbb{P}(E|A_i)\mathbb{P}(A_i)$

$= \sum_{\text{all } i} \frac{\mathbb{P}(E \cap A_i)}{\mathbb{P}(A_i)} \cdot \mathbb{P}(A_i)$ (definition of conditional probability)

$= \sum_{\text{all } i} \mathbb{P}(E \cap A_i)$

$= \mathbb{P}(E)$

The A_i partition Ω, so $E \cap A_i$ partition E. Then we just add up those probabilities.

Ability to add follows from the "countable additivity" axiom.

Bayes Rule

What do we know about Wonka Bars?

$$0.999 = \frac{\mathbb{P}(B|A) \cdot .010989}{.001}$$

Solving $\mathbb{P}(B|A) = \frac{1}{11}$, i.e. about 0.0909.

Only about a 10% chance that the bar has the golden ticket!

Wait a minute…

Willy Wonka has placed golden tickets on 0.1% of his Wonka Bars.
If the bar you weigh **does** have a golden ticket, the scale will alert you 99.9% of the time.
If the bar you weigh does not have a golden ticket, the scale will (falsely) alert you only 1% of the time.

That doesn't fit with many of our guesses. What's going on?

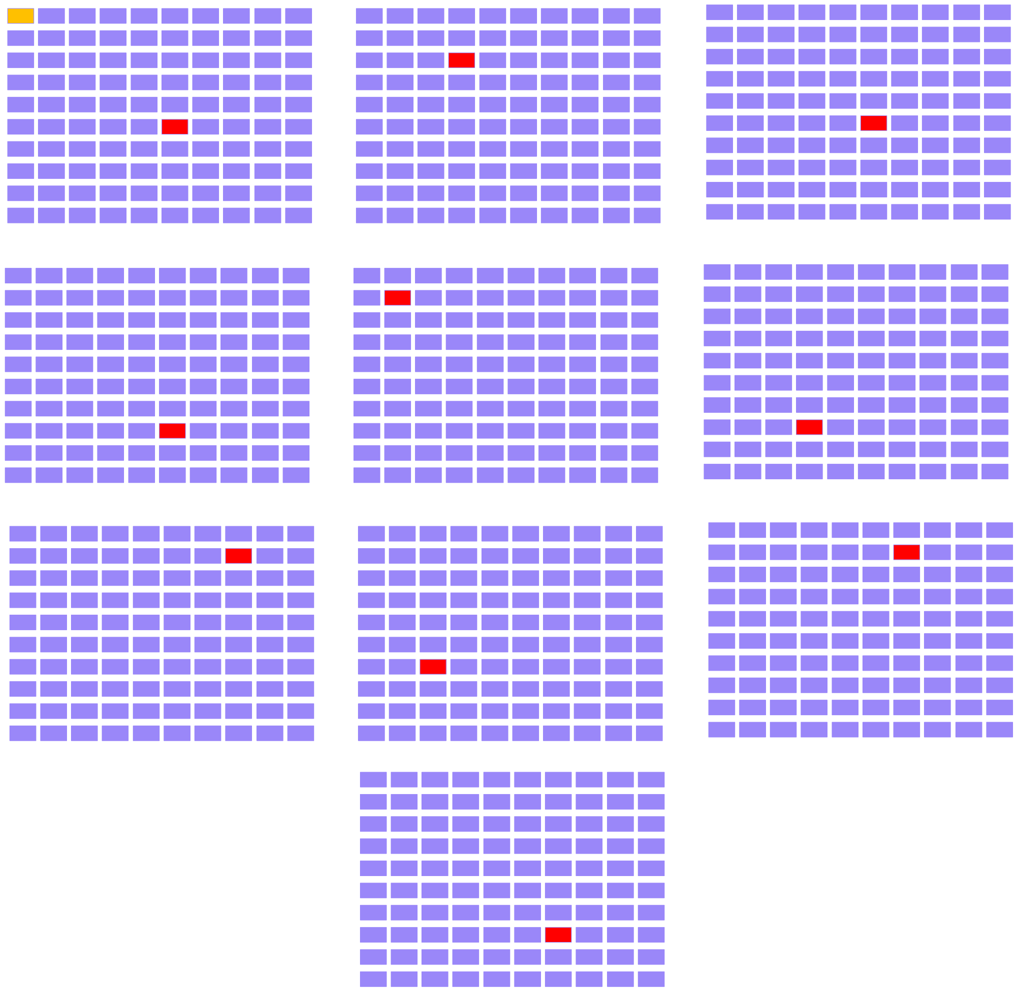
Instead of saying "we tested one and got a positive" imagine we tested 1000. **ABOUT** how many bars of each type are there?

(about) 1 with a golden ticket 999 without. Lets say those are exactly right.

Lets just say that one golden is truly found

(about) 1% of the 999 without would be a positive. Lets say it's exactly 10.

Visually



Gold bar is the one (true) golden ticket bar.
Purple bars don't have a ticket and tested negative.
Red bars don't have a ticket, but tested positive.

The test is, in a sense, doing really well.
It's almost always right.

The problem is it's also the case that the correct answer is almost always "no."

Updating Your Intuition

🔥 Take 1: The test is **actually good** and has VASTLY increased our belief that there IS a golden ticket when you get a positive result.

If we told you “your job is to find a Wonka Bar with a golden ticket” without the test, you have 1/1000 chance, with the test, you have (about) a 1/11 chance. That’s (almost) 100 times better!

This is actually a huge improvement!

Updating Your Intuition

🔥 Take 2: Humans are really bad at intuitively understanding very large or very small numbers.

When I hear “99% chance”, “99.9% chance”, “99.99% chance” they all go into my brain as “well that’s basically guaranteed” And then I forget how many 9’s there actually were.

But the number of 9s matters because they end up “cancelling” with the “number of 9’s” in the population that’s truly negative. We’ll talk about this a little more on Friday in the applications.

Updating Your Intuition

🔥 Take 3: View tests as updating your beliefs, not as revealing the truth.

Bayes' Rule says that $\mathbb{P}(B|A)$ has a factor of $\mathbb{P}(B)$ in it. You have to translate "The test says there's a golden ticket" to "the test says you should increase your estimate of the chances that you have a golden ticket."

A test takes you from your "prior" beliefs of the probability to your "posterior" beliefs.