Name: ____________________

Student ID: ____________________

CSE 312: Foundations of Computing II
Practice Final Exam

Instructions:

- Give your answers in the spaces provided on these sheets.
- Give a brief justification for each answer. Usually it is sufficient just to show the formula you are using and then substitute into the formula according to the problem details. Don't forget to define any events or random variables that you use, so that we know what your variables represent.
- If you continue an answer on the back of a page, be sure to indicate that on the front of the page.
- No calculators or other electronic devices allowed.

Reference Sheet

1. Formulas related to permutations and combinations.
   - $\binom{n}{k} = \frac{n!}{k!\cdot(n-k)!}$.
   - Binomial identity: $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^n y + \ldots + \binom{n}{n}y^n$.

2. Inclusion exclusion principle:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n| - |A_1 \cap A_2| - \cdots - |A_{n-1} \cap A_n| + \cdots .$$

3. Bayes' rule: $\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B|A)\cdot\mathbb{P}(A)}{\mathbb{P}(B)}$.

4. Chernoff-Hoeffding bound: If X is sum of independent Bernoulli's with mean μ, then

$$\mathbb{P}(|X-\mu| > \delta\mu) \leq 2e^{\delta^2\mu/(2+\delta)}.$$

5. Some standard discrete distributions:

distribution	**probability**	**expectation**	**variance**
Bernoulli	$\mathbb{P}(x=1)=p, \mathbb{P}(x=0)=1-p$	p	$p-p^2$
Binomial	$\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$	pn	$(p-p^2)n$
Geometric	$\mathbb{P}(X=k)=p(1-p)^{k-1}$	$1/p$	$\frac{1-p}{p^2}$
Poisson	$\mathbb{P}(X=k)=e^{-\lambda}\cdot\frac{\lambda^k}{k!}$	λ	λ

6. Some standard continuous distributions are shown below. Here $\Phi(x)$ is the cdf of the standard normal.

distribution	**pdf**	**cdf**	**expectation**	**variance**	**mgf**
Normal	$\frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$\Phi((x-\mu)/\sigma)$	μ	σ^2	$e^{t\mu}\cdot e^{t^2\sigma^2/2}$
Exponential	$\lambda e^{-\lambda t}$	$1-e^{-\lambda t}$	$1/\lambda$	$1/\lambda^2$	n/a

1. For each of the following assertions:

   - (3 points) State whether they are True or False.
   - (2 points) Briefly justify your answer.

   (a) The number of distinct rearrangements of the letters of the word BALLOON in which B and N occur together is $\frac{6!}{2!2!}$.
   False. Should be $2 \cdot \frac{6!}{2!2!}$

   (b) Let A and B be events in the same sample space. If $\mathbb{P}(A|B) = 1/2$, then $\mathbb{P}(A|B^c) = 1/2$.
   False. If $A \subset B$, then $\mathbb{P}(A|B^c) = 0$.

   (c) 50% of all rainy days start off cloudy. However, 40% of days start cloudy. If only 10% of the days are rainy, then the chance of rain during the day given that the morning is cloudy is 12.5%.
   True.
   By Bayes rule,

$$\begin{aligned}\mathbb{P}(\text{rain}|\text{cloud}) &= \frac{\mathbb{P}(\text{rain}) \cdot \mathbb{P}(\text{cloud}|\text{rain})}{\mathbb{P}(\text{cloud})} \\ &= \frac{0.1 \cdot 0.5}{0.4} = 0.125.\end{aligned}$$

   (d) If X is a random variable taking on non-negative integer values, and for each integer k, let E_k be the event that $X = k$. Then the set of events $E_0, E_1, \ldots$ form a partition of the probability space.
   True.

   (e) Suppose $X_1, \ldots, X_k$ are pairwise independent real valued random variables. Then

$\mathrm{Var}(X_1 + \ldots + X_k) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_k)$.

True.

$$\begin{aligned}
&\mathrm{Var}(X_1 + X_2 + \ldots + X_k) \\
&= \mathbb{E}\left((X_1 + \ldots + X_k)^2\right) - \mathbb{E}(X_1 + \ldots + X_k)^2 \\
&= \mathbb{E}\left(X_1^2 + \ldots + X_k^2 + 2X_1X_2 + \ldots + 2X_{k-1}X_k\right) \\
&- \left(\mathbb{E}\left(X_1^2\right) + \ldots + \mathbb{E}\left(X_k^2\right) + 2\mathbb{E}(X_1)\mathbb{E}(X_2) + \ldots + 2\mathbb{E}(X_{k-1})\mathbb{E}(X_k)\right) \\
&= \mathbb{E}\left(X_1^2\right) - \mathbb{E}(X_1)^2 + \mathbb{E}\left(X_2^2\right) - \mathbb{E}(X_2)^2 + \ldots + \mathbb{E}\left(X_k^2\right) - \mathbb{E}(X_k)^2 \\
&= \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_k),
\end{aligned}$$

where we used that $\mathbb{E}(X_iX_j) = \mathbb{E}(X_i)\mathbb{E}(X_j)$ since the variables are pairwise independent.

(f) Let $X \in \{1, 2, 3, 4\}$ be uniformly random and Y be independent and uniformly random in $\{1, 2, 3, 4, 5\}$. Then $\mathbb{P}(X = Y) = 1/5$.

True.

We have,

$$\begin{aligned}
\mathbb{P}(X = Y) &= \mathbb{P}(X = 1, Y = 1) + \mathbb{P}(X = 2, Y = 2) + \ldots + \mathbb{P}(X = 5, Y = 5) \\
&= \mathbb{P}(X = 1)\mathbb{P}(Y = 1) + \mathbb{P}(X = 2)\mathbb{P}(Y = 2) + \ldots + \mathbb{P}(X = 5)\mathbb{P}(Y = 5) \\
&= (1/4)(1/5) + \ldots + (1/4)(1/5) + 0(1/5) = 1/5.
\end{aligned}$$

(g) If X has a Poisson distribution with parameter 3, then $2X$ has Poisson distribution with paramter 6.

False. Writing down the expression for $\mathbb{P}(2X = k)$ implies that $2X$ is not Poisson.

(h) Let X be a real random variable with PDF f. Then the PDF of $2X$ is given by $\int_{-\infty}^{\infty} f(x')f(x - x')dx'$.

False. $\int_{-\infty}^{\infty} f(x')f(x - x')dx'$ is the PDF of $X_1 + X_2$ where X_1 and X_2 are independent and have the same PDF f. $\mathrm{Var}(2X) = 4\mathrm{Var}(X)$ and $\mathrm{Var}(X_1 + X_2) =$

$2\text{Var}(X)$. Hence, if $\text{Var}(X) \neq 0$, we can conclude that the PDF of $2X$ is different from $\int_{-\infty}^{\infty} f(x')f(x-x')dx'$.

(i) We have n balls and n bins. Each ball is thrown into a uniform random bin and each throw is independent of the other. Let X_i be 1 if the i'th bin is non-empty and 0 otherwise. Define $X = X_1 + \ldots + X_n$. We can now apply Chernoff's bound to say

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \delta \mathbb{E}(X)) \leq 2e^{-\frac{\delta^2 \mathbb{E}(X)}{2+\delta}}.$$

False. This is because the X_i's are not independent of each other.

(j) Let X and Y be random variables. If $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, then X and Y are not independent.

False. Independent X, Y satisfy $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

(k) Let X and Y be non-negative random variables. Then for every $a > 0$,

$$\mathbb{P}(X + Y \geq a) \leq \frac{\mathbb{E}(X) + \mathbb{E}(Y)}{a}.$$

True. This follows from Markov's inequality and linearity of expectation.

(l) If X is a random variable with $\mathbb{E}\left(e^{tX}\right) = e^{2t^2}$, then X must have a normal distribution.

True. e^{2t^2} is the Moment generating function of a normal distribution with mean 0 and variance 4. Since the same Moment generating function implies the same CDF, we can conclude that X has a normal distribution.

(m) Let X be a real random variable with CDF F. Then if $a < b$, we must have

$F(a) < F(b)$.

False. Could be equal.

(n) Suppose we are given two distributions on real numbers. For each n, suppose $X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_n$ are independent random variables, with X_i sampled according to the first distribution, and Y_i sampled according to the second distribution, for all i. Suppose the mean and standard deviations of the first distribution are μ_1, σ_1, and the corresponding parameters for the second distribution are μ_2, σ_2. Then for every number α, the limit

$$\lim_{n\to\infty} p\Big(\frac{X_1 + X_2 + \ldots + X_n + Y_1 + \ldots + Y_n - n(\mu_1 + \mu_2)}{\sqrt{n(\sigma_1^2 + \sigma_2^2)}} < \alpha\Big) = \Phi(\alpha),$$

where Φ is the cdf of the standard normal.

True. Follows from the Central Limit Theorem and the fact that $X_1 + X_2 + \ldots + X_n + Y_1 + \ldots + Y_n$ has mean $n(\mu_1 + \mu_2)$ and variance $n(\sigma_1^2 + \sigma_2^2)$.

Note. We haven't really covered the CLT for distinct random variables in this class.

2. You want to get rich quickly, so you buy 10,000 lottery tickets for \$1 each. Each ticket has probability 10^{-6} of winning \$6000, independently of the other tickets. Let random variable X be the number of winning lottery tickets among your 10,000.

   (a) (10 points) What is the probability that $X = k$?

   $\mathbb{P}(X = k) = \binom{10000}{k} 10^{-6k}(1 - 10^{-6})^{10000-k}$.

   (b) (10 points) Give the parameters for the Poisson random variable Y that approximates X well, and use the Poisson approximation to write a formula that should estimate the probability that you make a net profit with your 10,000 tickets.

   Y is a Poisson with parameter $\lambda = 10000 \cdot 10^{-6} = 10^{-2}$. Observe that we make a net profit if $X \geq 2$. We estimate $\mathbb{P}(X \geq 2)$ by $\mathbb{P}(Y \geq 2)$. We know that

$$\mathbb{P}(Y \geq 2) = 1 - \mathbb{P}(Y = 0) - \mathbb{P}(Y = 1) = 1 - e^{-\lambda} - e^{-\lambda}\lambda.$$

3. The time it takes to write a piece of software is modeled as a continuous random variable X from an unknown distribution. You would like to be able to guarantee a client that, with high probability, the software will be completed within 48 days. What is the best guarantee you could give under each of the following conditions? Justify your answer.

   (a) (6 points) You know that X has mean 20. Give an exact answer as a simplified fraction.

   The best guarantee is given by Markov's inequality:

   $$\mathbb{P}(X \geq 48) \leq 20/48.$$

   Therefore, $\mathbb{P}(X < 48) = 1 - \mathbb{P}(X \geq 48) \geq 1 - 20/48$.

   (b) (6 points) You know that X has mean 20 and variance 100. Give an exact answer as a simplified fraction.

   By Chebyshev's inequality,

   $$\mathbb{P}(|X - 20| \geq 28) \leq 100/28^2.$$

   Therefore, $\mathbb{P}(X < 48) = 1 - \mathbb{P}(X \geq 48) \geq 1 - 100/28^2$.

   (c) (8 points) You know that X is well approximated by the normal with mean 20 and standard deviation 100. Give your answer in terms of the cdf of the standard normal, $\Phi(x)$.

Note that $\frac{X-20}{100}$ is distributed according to a normal with mean 0 and variance 1. We have,

$$\mathbb{P}\left(X < 48\right) = \mathbb{P}\left(\frac{X-20}{100} < \frac{28}{100}\right) = \Phi(0.28).$$

4. You are playing a slot machine for which you must insert \$1 per play. It pays you back \$10 with probability 0.02, pays \$5 with probability 0.1, and pays nothing otherwise. Let the random variable X be your net gain in dollars on a single play. (Net gain includes the \$1 you pay to play. For example, if the machine pays you back \$10, your net gain is \$9.)

(a) (5 points) Compute $\mathrm{E}[X]$ exactly, with no rounding.

Note that

$$X = \begin{cases} 9, & \text{with probability } 0.02 \\ 4, & \text{with probability } 0.1 \\ -1, & \text{with probability } 0.88. \end{cases}$$

Therefore, $\mathrm{E}[X] = 9 \cdot 0.02 + 4 \cdot 0.1 + (-1) \cdot 0.88 = -0.3$.

(b) (7 points) Compute $\mathrm{Var}(X)$ exactly, with no rounding.

$\mathrm{Var}\,(X) = (9 + 0.3)^2 \cdot 0.02 + (4 + 0.3)^2 \cdot 0.1 + (-1 + 0.3)^2 \cdot 0.88 = 4.01$.

(c) (8 points) Let Y be your total net gain in 20 independent plays of this slot machine. Use the Central Limit Theorem to estimate the probability that you make a profit. Express your answer in terms of the cdf of the standard normal.

Y is a normal with mean -6 and variance 80.2. Therefore,

$$\mathbb{P}\,(Y > 0) = 1 - \mathbb{P}\,(Y \leq 0) = 1 - \mathbb{P}\left(\frac{Y + 6}{8.95} \leq 0.67\right) = 1 - \Phi(0.67).$$

5. You are in a disreputable casino, where you suspect they use a loaded 6-sided die. This loaded die rolls a 1 with probability 3θ (and rolls 2 with the same probability), rolls a 3 with probability 2θ (and rolls 4 with the same probability), and rolls a 5 with probability $\frac{1}{2} - 5\theta$ (and rolls 6 with the same probability), where $0 < \theta < 0.1$ is unknown. In order for you to beat the casino, it will help if you can estimate the value of θ. To do this, you record the outcomes of several independent rolls of the die as follows:

outcome	1	2	3	4	5	6
frequency	6	4	2	6	2	3

Find the maximum likelihood estimator $\hat{\theta}$ of θ. Show how to derive $\hat{\theta}$, and don't forget to show that it is a maximum.

(Hint to save you work: you will get the correct answer to this part even if you make the simplifying assumption that the first 6 rolls are 1, the next 4 rolls are 2, etc. Without this assumption, your likelihood function would have some complicated multinomial coefficient, but that factor has no dependence on θ so you can ignore it.)

The likelihood function is defined as

$$\begin{aligned} L(\theta) &= (3\theta)^6 \cdot (3\theta)^4 \cdot (2\theta)^2 \cdot (2\theta)^6 \cdot (1/2 - 5\theta)^2 \cdot (1/2 - 5\theta)^3 \\ &= (3\theta)^{10} \cdot (2\theta)^8 \cdot (1/2 - 5\theta)^5. \end{aligned}$$

We have to find the value of θ that maximizes L. It is sufficient to maximize

$$f(\theta) = 10 \log 3\theta + 8 \log 2\theta + 5 \log(1/2 - 5\theta).$$

We know that

$$f'(\theta) = \frac{10}{\theta} + \frac{8}{\theta} - \frac{25}{1/2 - 5\theta} = \frac{18}{\theta} - \frac{50}{1 - 10\theta}.$$

Solving for $f'(\theta) = 0$, we get that $\theta = \frac{9}{115}$.

To verify that this is the maximizer, we now compute $f''(\theta)$.

$$f''(\theta) = -\frac{18}{\theta^2} - \frac{500}{(1 - 10\theta)^2}.$$

Since $0 < \theta < 0.1$, we can conclude that $f''(\theta) \leq 0$, implying that the function is concave and $\theta = \frac{9}{115}$ is the maximizer.