Reference Sheet: Counting, Discrete Probability

Theorem: Binomial Theorem

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Theorem: Principle of Inclusion-Exclusion (PIE)

2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
k events: singles - doubles + triples - quads + ...

Theorem: Pigeonhole Principle

If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

Definition: Key Probability Definitions

The **sample space** is the set Ω of all possible outcomes of an experiment.
An **event** is any subset $E \subseteq \Omega$.
Events E and F are **mutually exclusive** if $E \cap F = \emptyset$.

Definition: Probability space

A *probablity space* is a pair $(\Omega, \mathbb{P})$, where Ω is the sample space
$\mathbb{P} : \Omega \to [0,1]$ is a *probability measure* such that $\sum_{x \in \Omega} \mathbb{P}(x) = 1$.
The probability of an event $E \subseteq \Omega$ is $\mathbb{P}(E) = \sum_{x \in E} \mathbb{P}(x)$.

Definition: Conditional Probability

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

Theorem: Bayes Theorem

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[B \mid A]\,\mathbb{P}[A]}{\mathbb{P}[B]}$$

Definition: Partition

Non-empty events $E_1, \ldots, E_n$ **partition** the sample space Ω if:

- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ (none of them overlap)

Theorem: Law of Total Probability (LTP)

If events $E_1, \ldots, E_n$ partition Ω, then for any event F:

$$\mathbb{P}[F] = \sum_{i=1}^{n} \mathbb{P}[F \cap E_i] = \sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]$$

Theorem: Bayes Theorem with LTP

Let events $E_1, \ldots, E_n$ partition the sample space Ω, and let F be another event. Then:

$$\mathbb{P}[E_1 \mid F] = \frac{\mathbb{P}[F \mid E_1]\,\mathbb{P}[E_1]}{\sum_{i=1}^{n} \mathbb{P}[F \mid E_i]\,\mathbb{P}[E_i]}$$

Definition: Independence (Events)

A and B are **independent** if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$
2. $\mathbb{P}[A \mid B] = \mathbb{P}[A]$
3. $\mathbb{P}[B \mid A] = \mathbb{P}[B]$

Theorem: Chain Rule

Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:
$\mathbb{P}[A_1 \cap \cdots \cap A_n] =$
$\mathbb{P}[A_1]\,\mathbb{P}[A_2 \mid A_1]\,\mathbb{P}[A_3 \mid A_1 \cap A_2] \cdots \mathbb{P}[A_n \mid A_1 \cap \cdots \cap A_{n-1}]$

Definition: Mutual Independence (Events)

We say n events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \mathbb{P}[A_i]$$

This equation is actually representing 2^n equations since there are 2^n subsets of $[n]$.

Definition: Conditional Independence

A and B are **conditionally independent given an event** C if any of the following equivalent statements hold:
1. $\mathbb{P}[A \cap B \mid C] = \mathbb{P}[A \mid C]\,\mathbb{P}[B \mid C]$
2. $\mathbb{P}[A \mid B \cap C] = \mathbb{P}[A \mid C]$
3. $\mathbb{P}[B \mid A \cap C] = \mathbb{P}[B \mid C]$

Definition: Random Variable (RV)

A random variable X is a function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values X can take on is its **range/support**, denoted Ω_X.

Definition: Probability Mass Function (PMF)

For a discrete RV X, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0,1]$ where: $p_X(k) = \mathbb{P}[X = k]$.

Definition: Expectation

The **expectation** of a discrete RV X is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

Theorem: Linearity of Expectation (LoE)

For any random variables X, Y (possibly dependent):
$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Theorem: Law of the Unconscious Statistician (LOTUS)

For a discrete RV X and function g, $\mathbb{E}[g(X)] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

Definition: Variance

$\text{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

Theorem: Property of Variance

$\text{Var}(aX + b) = a^2 \text{Var}(X)$.

Definition: Independence (Random Variables)

Random variables X and Y are **independent** if for *all* $x \in \Omega_X$ and all $y \in \Omega_Y$:
$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x] \cdot \mathbb{P}[Y = y].$$

Theorem: Variance Adds for Independent RVs

If X, Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Definition: Standard Deviation (SD)

$\sigma_X = \sqrt{\text{Var}(X)}$.

Reference: Continuous and Multivariate Probability

Definition: Cumulative Distribution Function (CDF)

The **cumulative distribution function (CDF)** of ANY random variable is $F_X(t) = \mathbb{P}[X \leq t]$.
If X is a *continuous* RV, $F_X(t) = \mathbb{P}[X \leq t] = \int_{-\infty}^{t} f_X(w)\, dw$.

Theorem: Multiplicativity of expectation

For any independent random variables X, Y:

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Definition: Expectation (Continuous)

The **expectation** of a continuous RV X is:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx.$$

Theorem: Law of the Unconscious Statistician (LOTUS)

For a continuous RV X: $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$.

Definition: Independent and Identically Distributed (i.i.d.)

We say $X_1, \ldots, X_n$ are said to be **independent and identically distributed (i.i.d.)** if all the X_i's are independent of each other, and have the same distribution (PMF for discrete RVs, or CDF for continuous RVs).

Definition: Joint PMFs

The joint PMF of discrete RVs X and Y is:

$$p_{X,Y}(a,b) = \mathbb{P}[X = a, Y = b]$$

Their joint range is

$$\Omega_{X,Y} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that $\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1$.

Definition: Joint PDFs

The joint PDF of continuous RVs X and Y is:

$$f_{X,Y}(a,b) \geq 0$$

Their joint range is

$$\Omega_{X,Y} = \{(c,d) : f_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv = 1$.

Definition: Marginal PMFs

Let X, Y be discrete random variables. The marginal PMF of X is: $p_X(a) = \sum_{b\in\Omega_Y} p_{X,Y}(a,b)$.

Definition: Marginal PDFs

Let X, Y be continuous random variables. The marginal PDF of X is: $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$.

Definition: Independence of RVs (Continuous)

Continuous RVs X, Y are independent, written $X \perp Y$, if for all $x \in \Omega_X$ and $y \in \Omega_Y$,

$$f_{X,Y}(x,y) = f_X(x) f_Y(y).$$

Definition: Conditional Expectation

If X is discrete (and Y is either discrete or continuous), then we define the conditional expectation of $g(X)$ given (the event that) $Y = y$ as:

$$\mathbb{E}[\,g(X) \mid Y = y\,] = \sum_{x\in\Omega_X} g(x)\,\mathbb{P}(X = x \mid Y = y)$$

If X is continuous (and Y is either discrete or continuous), then

$$\mathbb{E}[\,g(X) \mid Y = y\,] = \int_{-\infty}^{\infty} g(x)\,\frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

Theorem: Law of Total Expectation (LTE)

Let X, Y be jointly distributed random variables.

If Y is discrete (and X is either discrete or continuous), then:

$$\mathbb{E}[\,g(X)\,] = \sum_{y\in\Omega_Y} \mathbb{E}[\,g(X) \mid Y = y\,]\; p_Y(y)$$

If Y is continuous (and X is either discrete or continuous), then

$$\mathbb{E}[\,g(X)\,] = \int_{-\infty}^{\infty} \mathbb{E}[\,g(X) \mid Y = y\,]\; f_Y(y)\,dy$$

Reference: Tail Bounds

Theorem: Markov's Inequality

Let $X \geq 0$ be a **non-negative** RV, and let $k > 0$. Then:

$$\mathbb{P}[X \geq k] \leq \frac{\mathbb{E}[X]}{k}$$

.

Theorem: Chebyshev's Inequality

Let X be any RV with expected value $\mu = \mathbb{E}[X]$ and finite variance $\text{Var}(X)$. Then, for any real number $\alpha > 0$. Then,

$$\mathbb{P}[|X - \mu| \geq \alpha] \leq \frac{\text{Var}(X)}{\alpha^2}$$

.

Theorem: Chernoff Bound

Let $X = X_1 + X_2 + \ldots + X_n$, where $X_1, X_2, \ldots, X_n$ are independent random variables, each taking values in $[0,1]$. Also, let $\mu = \mathbb{E}[X]$.
For any $1 > \delta > 0$:

$$\mathbb{P}(X \geq (1+\delta)\mu) \leq \exp\left(-\delta^2\mu/3\right)$$

$$\mathbb{P}(X \leq (1-\delta)\mu) \leq \exp\left(-\delta^2\mu/2\right)$$

Theorem: The Union Bound

Let $E_1, E_2, \ldots, E_n$ be a collection of events. Then:

$$\mathbb{P}\left[\bigcup_{i=1}^{n} E_i\right] \leq \sum_{i=1}^{n} \mathbb{P}[E_i]$$

Reference: Zoo

Definition: Bernoulli/Indicator Random Variable

$X \sim \text{Bernoulli}(p)$ (Ber(p) for short) iff X has PMF:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$ and $\text{Var}(X) = p(1-p)$.

Definition: Binomial Random Variable

$X \sim \text{Binomial}(n, p)$ (Bin(n, p) for short) iff X has PMF

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k \in \Omega_X = \{0, 1, \dots, n\}$$

$\mathbb{E}[X] = np$ and $\text{Var}(X) = np(1-p)$.

Definition: Uniform Random Variable (Discrete)

$X \sim \text{Uniform}(a, b)$ (Unif(a, b) for short), for integers $a \leq b$, iff X has PMF:

$$p_X(k) = \frac{1}{b-a+1}, \quad k \in \Omega_X = \{a, a+1, \dots, b\}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$.

Definition: Geometric Random Variable

$X \sim \text{Geometric}(p)$ (Geo(p) for short) iff X has PMF:

$$p_X(k) = (1-p)^{k-1} p, \quad k \in \Omega_X = \{1, 2, 3, \dots\}$$

$\mathbb{E}[X] = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$.

Definition: Poisson Random Variable

$X \sim \text{Poisson}(\lambda)$ (Poi(λ) for short) iff X has PMF:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \Omega_X = \{0, 1, 2, \dots\}$$

$\mathbb{E}[X] = \lambda$ and $\text{Var}(X) = \lambda$. If $X_1, \dots, X_n$ are independent Poisson RV's, where $X_i \sim \text{Poi}(\lambda_i)$, then $X = X_1 + \dots + X_n \sim \text{Poi}(\lambda_1 + \dots + \lambda_n)$.

Definition: Uniform Random Variable (Continuous)

$X \sim \text{Uniform}(a, b)$ (Unif(a, b) for short) iff X has PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in \Omega_X = [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$.

Definition: Exponential Random Variable

$X \sim \text{Exponential}(\lambda)$ (Exp(λ) for short) iff X has PDF:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \in \Omega_X = [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$.

$F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$.

Definition: Normal (Gaussian, "bell curve") Random Variable

$X \sim \mathcal{N}(\mu, \sigma^2)$ iff X has PDF:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \Omega_X = \mathbb{R}$$

$\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

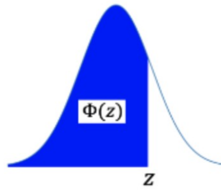
Theorem: Closure of the Normal Under Scale and Shift

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.
In particular, we can always scale/shift to get the standard Normal: $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$.

Theorem: Closure of the Normal Under Addition

If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ are *independent*, then

$$aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$$



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999