Homework 8: MLEs and Multiple Variables Review

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator. For each problem, make sure to explicitly define all random variables you use, and be clear about how they are related to each other using proper notation (conditionals, summations, etc.).

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Instructions as to how to upload your solutions to gradescope are on the course web page.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 8". (Instructions as to how to upload your solutions to gradescope are on the course web page.) The use of latex is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

Due Date: This assignment is due Wednesday May 29 at 11:59 PM.
You will submit the written problems as a PDF to gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways). The coding problem will also be submitted to gradescope.

0. Extra Instructions :o

For calculations that require evaluating integrals (unless we indicate otherwise), you must

(a) Show the integral to evaluate (e.g., $\int_0^2 z \cdot 2dz$)

(b) Show an antiderivative and the values to evaluate at (e.g., $z^2|_0^2$)

(c) Plug in the values and simplify (e.g., $2^2 - 0^2 = 4$)

This is not a problem, so nothing needs to be submitted here.

1. Some Multiple Choice Questions [8 points]

This problem will be autograded on gradescope. We have some multiple choice questions related to topics from previous weeks, based on questions we've been getting in office hours.

If you have a solid understanding of these topics, the multiple choice questions should be quick—if they aren't, you might want to review those topics or ask us questions at office hours.

2. Maximally Liked Dog Adopter [15 points]

Suppose UW professors Robbie, Anna, and Paul want to adopt some dogs and enter a dog shelter together. Out of sheer coincidence, all three professors immediately fall in love with the same set of n dogs. They each wish to adopt all n dogs, but out of fairness, agree to let the dogs decide who they want to go home with.

Suppose that each dog independently chooses to go home with Robbie with probability θ_R, Anna with probability θ_A, and Paul with probability $1 - \theta_R - \theta_A$. (Thus, $0 \leq \theta_R + \theta_A \leq 1$). The parameters θ_R, θ_A are unknown. Suppose that $x_1, ..., x_n$ are n independent, identically distributed samples from this distribution of professor choices. Let n_R be the number of x_i's equal to Robbie, let n_A be the number of x_i's equal to Anna, and let n_P be the number of x_i's equal to Paul. What are the maximum likelihood estimates for θ_R and θ_A in terms of n_R, n_A, and n_P?

In doing this problem, you do not need to do a second-derivative test (or any other test) to confirm you have a maximizer. You may assume any critical point you find is a maximizer.

For this problem, except where noted, all of your work must be understandable without reference to any calculator (including Wolfram-Alpha). You may check your answers using calculators, but your explanation may not rely on them.

(a) Write the Likelihood function. [3 points]

(b) Write the log-likelihood. [2 points]

(c) Give two equations describing the maximizer of the log-likelihood. Be sure to show work (e.g., taking of partial derivatives) on how you got these equations. [8 points]

(d) You should now have a system of two (linear) equations in two unknowns. Find the solution to this system, and use it to write the MLEs for θ_R and θ_A. You do **not** have to show your work for this part, and may use online algebra solvers if you wish. [2 points]

3. More MLE [18 points]

Suppose you have a density

$$f_Y(y) = \begin{cases} \frac{-|y|}{\theta^2} + \frac{1}{\theta} & \text{if } -\theta \leq y \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

Let $y_1, y_2, \ldots, y_n$ be independent draws of a random variable from this distribution for some θ. **For this problem, *starting with part (b)*, all of your work must be understandable without reference to any calculator (including Wolfram-Alpha). You may check your answers using calculators, but your explanation may not rely on them.**

(a) Graph this density function for a few values of θ. Describe in a sentence or two what the shape is. The main goal of this part is for you to get intuition about the problem - you don't need to include the graphs. [2 points]

(b) Write down the likelihood function for this problem; be careful. You should probably come back to this question after doing part c to make sure you've handled edge cases. [3 points]

(c) Suppose you look at a value of θ such that $|y_1| \geq \theta$. What is the likelihood $\mathcal{L}(y_1, ..., y_n; \theta)$ in this case? Why? [3 points]

(d) In order to finish the problem without brutal algebra, we're going to assume that we've gotten exactly two independent samples y_1, y_2 and, moreover, assume that $y_1 = -y_2$.[1] Write the likelihood function under this extra assumption. [3 points]

(e) Find the maximum likelihood estimator $\hat{\theta}$ under our extra assumption in part d. (be sure to think about your answer in (c)). You may skip the step of using the second derivative test to verify that your critical point is a maximizer (but you still must consider the piecewise part of the definition of the likelihood function in (b)). [7 points]

[1] This is not a normal assumption to make for an MLE calculation, but the algebra is much worse in the general case, and this very simplified case will still be good practice.

4. Conditional Expectations [15 points]

Let X and Y be continuous random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{27}x^2 & 0 \leq x \leq y \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $f_{Y|X}(y|x)$?

(b) Determine $\mathbb{E}[Y|X = x]$.

(c) What is $\mathbb{E}[Y]$?

5. Unbiased Estimators [5 points]

Let $X_1, X_2, ..., X_n$ be random samples drawn from the continuous distribution Uniform$(0, \theta)$, for $\theta > 0$. We stumble upon a possible estimator $\hat{\theta}$ for θ, where $\hat{\theta} = 2 \cdot \sum_{i=1}^{n} X_i/n$. Is $\hat{\theta}$ an unbiased estimator of θ?