Homework 5: Discrete and Continuous

Version 2: Uploaded 5/3 9 AM; we added the feedback question.

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Remember that you must tag your written problems on Gradescope.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 5 [Written]". (Instructions as to how to upload your solutions to gradescope are on the course web page.)

Due Date: This assignment is due Wednesday May 8 at 11:59 PM.
You will submit the written problems as a PDF to gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Dice Try! [15 points]

You are playing a game that uses a fair 10-sided die whose faces are numbered 1, 2, ..., 10. The value of a roll is the number showing on the top of the die when it comes to rest. **Give all answers as simplified fractions.**

(a) Let X be the value of one roll of the die. Compute $\mathbb{E}[X]$ and Var (X). For this problem, you can either calculate from first principles (i.e. definitions and theorems) or describe which variable from the zoo is appropriate and what values you get as a result.

(b) Let Y be the sum of the values of 6 independent rolls of the die. Compute $\mathbb{E}[Y]$ and Var (Y). Use independence, and state **precisely** where in your computation you are using it.

(c) Let Z be the **average** of the values of 6 independent rolls of the die. Compute $\mathbb{E}[Z]$ and Var (Z). Use independence, and state **precisely** where in your computation you are using it.

2. Volleyball [10 points]

A set in volleyball ends when a team has:

- at least 25 points *and*
- at least 2 more points than their opponent.

So, for example, a set at 25-24 is not over (no one has a two point lead), but a set at 27-25 is over.

Suppose a set is tied 24-24, and each point is won by your team (independently) with probability p. What is the expected number of points played before one team or the other wins the set? (Hint: You could calculate this with an infinite sum, in which case you may use wolframalpha to find a closed form. But a clever use of a variable from the zoo will save you quite a bit of work, and make it so an infinite summation is unnecessary.)

3. PDF and CDF [10 points]

For this exercise, give exact answers involving simplified fractions as necessary. If you need to evaluate a definite integral, be sure to list the integral, the antiderivative, and what you get when plugging in the limits of integration as part of your work.

Consider the following PDF:

$$f_X(x) = \begin{cases} m(2 - 8x^2) & \text{if } -\frac{1}{4} \le x \le \frac{1}{4} \\ 0 & \text{otherwise} \end{cases}$$

Where m is a constant.

(a) What value of m makes the PDF valid?

(b) What is $F_X(k)$?

4. The Classic Flea Problem [16 points]

A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius b. At this moment, it is equally likely to be at any point within the ball. Let X be the distance of the flea from the center of the ball. For X, find

(a) the cumulative distribution function F. [5 points]

(b) the probability density function f. [5 points]

(c) the expected value. [4 points]

(d) the variance. [4 points]

Reminder: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

5. Exponential Darts [12 points]

You throw a dart at a circular target of radius r. Let X be the distance of your dart's hit from the center of the target. You have improved and your aim is such that $X \sim \text{Exponential}(5/r)$. (Note that it is possible for the dart to completely miss the target.)

(a) As a function of r, determine the value m such that $\mathbb{P}(X < m) = \mathbb{P}(X > m)$. [6 points]

(b) For $r = 15$, give the value of m to 3 decimal places. [2 points]

(c) What is the probability that you miss the target completely? Give your answer to 4 decimal places. [4 points]

6. Feedback [1 point]

Answer these questions on the separate Gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?