Homework 2: More Counting and Probability

For each problem, remember you must briefly explain/justify how you obtained your answer, as correct answers without an explanation will not receive full credit. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not read the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 2 [Written]". (Instructions as to how to upload your solutions to Gradescope are on the course web page.) The use of $\LaTeX$ is *highly recommended*. (Note that if you want to hand-write your solutions, you'll need to scan them. We will take off points for hand-written solutions that are difficult to read due to poor handwriting and neatness.)

You will submit the written problems as a PDF to Gradescope. Please put each numbered problem on its own page of the pdf (this will make selecting pages easier when you submit), and ensure that your pdfs are oriented correctly (e.g. not upside-down or sideways).

Version 2: Updated 4/8 at 11 AM. We updated the hint for Question 1b.
Due Date: This assignment is due Wednesday April 10 at 11:59 PM.

Collaboration: Please read the full collaboration policy. If you work with others (and you should!), you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

1. Combinatorial Identities [16 points]

Prove each of the following identities using a *combinatorial argument* (i.e., an argument that counts two different ways); an algebraic solution will be marked substantially incorrect.

For the purposes of these problems, using commutativity of multiplication and addition (i.e. $ab = ba$, $a + b = b + a$), and distributivity/factoring ($a(b + c) = ab + ac$) are allowed as part of a combinatorial argument. Any other algebra facts (e.g. Pascal's Rule about combinations, the definition of combinations/permutations in terms of factorials, canceling numbers that appear in numerators and denominators) would make it an algebraic solution, not a combinatorial one.

(a) $\sum_{i=0}^{n} \binom{n}{i}\binom{m}{i} = \binom{n+m}{n}$. You may assume that $m \geq n \geq 0$.
Hint: Start with the right-hand side and imagine you are choosing a team of n TAs for Fall, from a group consisting of the n current (Spring) TAs and m new applicants.

(b) $\sum_{i=n}^{m} \binom{m}{i}\binom{i}{n} = \binom{m}{n} 2^{m-n}$. Assume that $m \geq n \geq 0$.
Hint: Think about choosing a committee of varying size from m people with a subcommittee of fixed size.

2. Pikachu Greetings [10 points]

When a group of pikachu meet, they may introduce themselves by shaking tails. In a large group of pikachu, there won't be time for every pikachu to shake tails with every other, though the same pikachu may shake tails with multiple others.

Suppose you have a group of n pikachu. Assign each pikachu a "friendliness score", which is the number of other pikachu it shakes tails with.

Use the pigeonhole principle to show the following claim:

In a group of n pikachu, there are at least two with the same friendliness score.

You can assume "shaking tails" is symmetric (if A shakes tails with B, then B also does with A), and a Pikachu cannot shake tails with itself.
When using the pigeonhole principle, be sure to mention what the pigeons and pigeonholes are.

Hint: You may want to break into cases; think about what happens if a pikachu has a friendliness score of 0.

3. Stuff into stuff [12 points]

(a) We have 30 (distinguishable) people and 50 (distinguishable) rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 50 rooms.)

(b) We have 25 identical (indistinguishable) apples. How many different ways are there to place the apples into 20 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

(c) We have 35 identical (indistinguishable) apples. How many different ways are there to place the apples into 10 (distinguishable) boxes, if each box is required to have at least two apples in it?

4. Sample Spaces and Probabilities [18 points]

For each of the following scenarios first describe the sample space and indicate how big it is (i.e., what its cardinality is) and then answer the question.

(a) You flip a fair coin 40 times. What is the probability of exactly 15 heads?

(b) You roll 2 fair 6-sided dice, one red and one blue. What is the probability that the sum of the two values showing is 5?

(c) You are given a random 5 card poker hand (selected from a single deck). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?

(d) 20 labeled balls are placed into 15 labeled bins (each placement of all the bals into bins is equally likely). What is the probability that bin 1 contains exactly 3 balls?

(e) There are 24 psychiatrists and 16 psychologists attending a certain conference. Three of these 40 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?

(f) You buy 12 cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. For this problem, each distinguishable collection of cupcakes is equally likely (for example, the probability of getting 12 chocolate cupcakes is equal to the probability of getting 5 chocolate, 3 vanilla, and 4 caramel cupcakes in any order). What is the probability that you have at least one of each type?

5. Miscounting [14 points]

Consider the question: How many **7-card** poker hands (order doesn't matter) are there that contain at least two 3-of-a-kinds (3-of-a-kind means three cards of the same value). For example, this would be a valid hand: ace of hearts, ace of diamonds, ace of spaces, 7 of clubs, 7 of spades, 7 of hearts, and queen of clubs. (Note that a hand consisting of all 4 aces and three of the 7s is also valid.)

Here is how we might compute this:

> To compute the number of hands, apply the product rule using the following sequential process. First pick two ranks that have a 3-of-a-kind (e.g. ace and 7 in the example above). For the lower rank of these, pick the suits of the three cards. Then for the higher rank of these, pick the suits of the three cards. Then out of the remaining $52 - 6 = 46$ cards, pick one. Therefore there are
>
> $$\binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{3} \cdot \binom{46}{1}$$
>
> hands.

In this problem, you will find what is wrong with this solution.

(a) Is there overcounting in the solution? That is, is there a hand that can be produced by multiple outcomes of the sequential process? If there is, give one concrete example of such a hand and two outcomes of the process that produce it. If there is not, briefly (1-2 sentences) explain why there isn't.

(b) Is there undercounting in the problem? That is, is there a hand that cannot be produced by any outcomes of the sequential process? If there is, give one concrete example of such a hand and briefly explain why no outcome produces it. If there is no such hand, briefly explain why all hands are produced at least once.

(c) Correct the calculation – in this part you should produce a correct overall formula by subtracting/dividing out any errors that would fit in (a) and adding/multiplying in any errors that would fit in (b).

(d) Find the answer differently – take a different approach to counting this problem (e.g. use a different sequential process). Verify that you get the same number (via a different formula) than the last part.

6. Real-World: Is this a pigeon? [6 points]

The pigeonhole principle is surprisingly powerful and can be used to prove a variety of things - find one application of the principle that interests you (it can be CS-related or not, serious or not; the only requirements are that (1) you must find it interesting and (2) we shouldn't have proved it already in this course) and explain the proof to us.

(a) Provide your source and some background for the application. [2 points]

(b) Relate your proof to the theorem definition from lecture - what are the pigeons and what are the pigeonholes? [2 points]

(c) What does the theorem tell us about them? [4 points]

6.1. Some suggestions

We hope you'll think of something on your own! If you can't, here are some ideas that you can use:

- Husky Den Food Court options
- UW Course Conflicts (e.g., based on times of day; you can't do number of courses you take per quarter since we did that one in class).
- Olympic Sports

6.2. Rubric information

These 'real world' problems are a bit different from other problems! Because of that we're showing you what our rubric is going to be for this first one so you can see our expectations.

(a) There should be enough information here that we can check that you have a correct application in the next part. This might be link(s) to website(s) where you got the information, or it might be you telling us how you found something (e.g., "I went to the foodcourt and counted options"). If it's something we might not understand (e.g., a hobby you think the grader might not share), explain enough for us to know what you're talking about.

(b) It needs to be clear what set your pigeons are and what your pigeonholes are.

(c) We need to understand the conclusion you're making, and it needs to be an accurate conclusion from part (b).

Below are two sample solutions; the first would get credit for each part; the second would get no credit for each part.

6.3. Sample Solution - Credit for each part

(a) Source: https://www.athletic.net/TrackAndField/meet/494374/results/m/1/1mile

Background: Nike Cross Nationals (NXN) is the defacto championship for high school running. There were 57 participants in the mile race this year. The range of the times (only counting whole seconds) was 27 seconds, with the champion running a 4:02 and the last-place finisher clocking in at 4:28.

(b) Let the pigeons be runners. Let the pigeonholes be mile times.

(c) Since there are 57 runners, $n = 57$. Since there were 27 different finishing times, $k = 27$. We observe that $\lceil \frac{57}{27} \rceil = 3$. Therefore, the pigeonhole principle allows us to conclude that at least one finishing time was attained by at least three people. This is confirmed by the results. In fact, 5 people finished with a 4:13. That must have been quite a fun race to watch.

6.4. Sample Solution - No Credit for each part

(a) Source: the internet

Background: Nike Cross Nationals (NXN)

(b) There are runners and mile times.

(c) Some runners will end at the same time.

7. Feedback [1 point]

Answer these questions on the separate Gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?

- Do you have any thoughts about specifically “Real-World: Is this a pigeon?” that you would like to share with us?
- Any other feedback for us?