CSE 312

Foundations of Computing II

Lecture 18: Joint Distributions

Midterm

- We are finishing grading today – for now focus on PSet 5
- I will spend a few minutes talking about midterm results on Friday

Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
- Conditional Expectation and Law of Total Expectation

Mostly formalism helping us with multiple random variables

Why joint distributions?

- Given all of its user's ratings for different movies, and any preferences you have expressed, Netflix wants to recommend a new movie for you.

- Given a large amount of medical data correlating symptoms and personal history with diseases, predict what is ailing a person with a particular medical history and set of symptoms.

- Given current traffic, pedestrian locations, weather, lights, etc. decide whether a self-driving car should slow down or come to a stop

Review Cartesian Product

Definition. Let A and B be sets. The **Cartesian product** of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example.

$$\{1,2,3\} \times \{4,5\} = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)

Joint PMFs and Joint Range

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = P(X = a, Y = b)$$

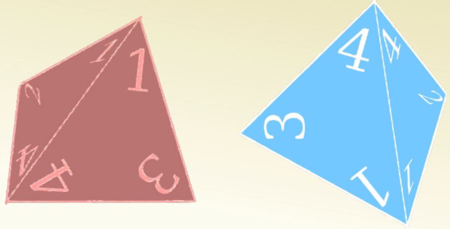
Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega_{X,Y} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

Note that

$$\sum_{(s,t) \in \Omega_{X,Y}} p_{X,Y}(s,t) = 1$$

Example – Weird Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die.

$\Omega_X = \{1,2,3,4\}$ and $\Omega_Y = \{1,2,3,4\}$

In this problem, the joint PMF is if

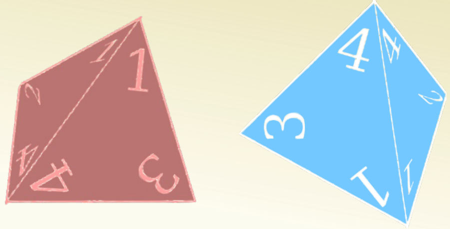
$$p_{X,Y}(x,y) = \begin{cases} 1/16 & \text{if } x, y \in \Omega_{X,Y} \\ 0 & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

and the joint range is (since all combinations have non-zero probability)

$\Omega_{X,Y} = \Omega_X \times \Omega_Y$

Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

$$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \Omega_U \times \Omega_W$$

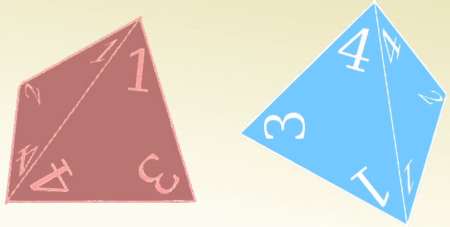
Poll: pollev.com/stefanotessaro617

What is $p_{U,W}(1, 3) = P(U = 1, W = 3)$?

a. $1/16$
b. $2/16$
c. $1/2$
d. Not sure

U\W	1	2	3	4
1				
2				
3				
4				

Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega_U = \{1,2,3,4\}$ and $\Omega_W = \{1,2,3,4\}$

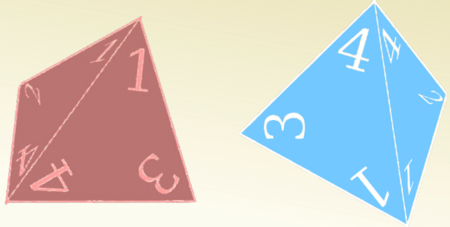
$$\Omega_{U,W} = \{(u, w) \in \Omega_U \times \Omega_W : u \leq w\} \neq \Omega_U \times \Omega_W$$

The joint PMF $p_{U,W}(u, w) = P(U = u, W = w)$ is

$$p_{U,W}(u, w) = \begin{cases} 2/16 & \text{if } (u, w) \in \Omega_U \times \Omega_W \text{ where } w > u \\ 1/16 & \text{if } (u, w) \in \Omega_U \times \Omega_W \text{ where } w = u \\ 0 & \text{otherwise} \end{cases}$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example – Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $P(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

Just apply LTP over the possible values of W:

$$p_U(1) = 7/16$$

$$p_U(2) = 5/16$$

$$p_U(3) = 3/16$$

$$p_U(4) = 1/16$$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Marginal PMF

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **marginal PMF** of X

$$p_X(a) = \sum_{b \in \Omega_Y} p_{X,Y}(a,b)$$

Similarly, $p_Y(b) = \sum_{a \in \Omega_X} p_{X,Y}(a,b)$

Continuous distributions on $\mathbb{R}\times\mathbb{R}$

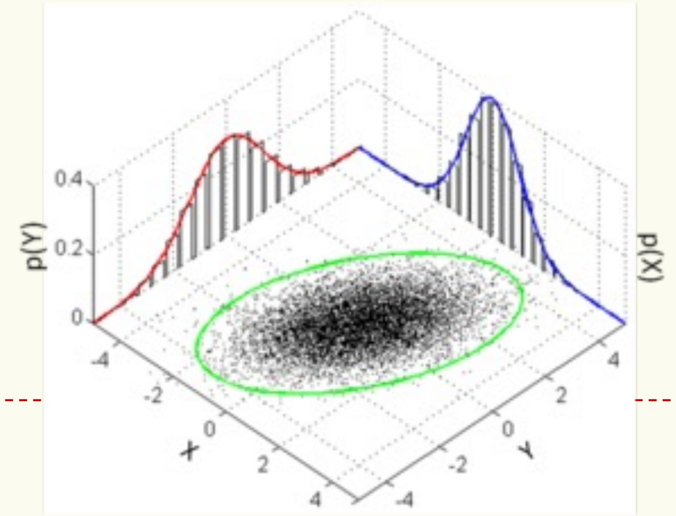
Definition. The **joint probability density function (PDF)** of continuous random variables X and Y is a function $f_{X,Y}$ defined on $\mathbb{R}\times\mathbb{R}$ such that

- $f_{X,Y}(x,y) \geq 0$ for all $x, y \in \mathbb{R}$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\mathrm{d}x\ \mathrm{d}y = 1$

for $A \subseteq \mathbb{R}\times\mathbb{R}$ the probability that $(X,Y) \in A$ is $\iint_A f_{X,Y}(x,y)\ \mathrm{d}x\mathrm{d}y$

The **(marginal) PDFs** f_X and f_Y are given by

- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}y$
- $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ \mathrm{d}x$



Independence and joint distributions

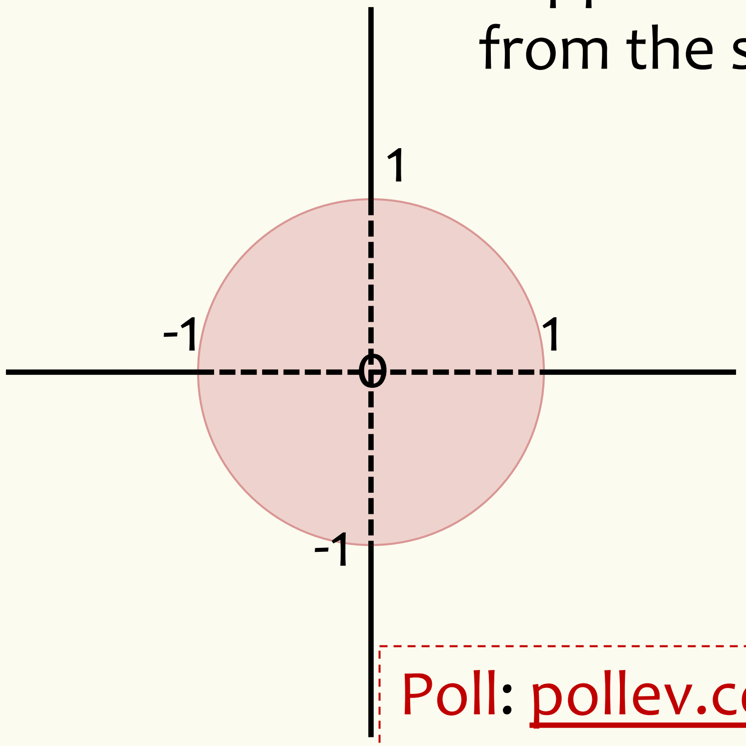
Definition. Discrete random variables X and Y are **independent** iff

- $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$ for all $x \in \Omega_X, y \in \Omega_Y$

Definition. Continuous random variables X and Y are **independent** iff

- $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ for all $x, y \in \mathbb{R}$

Example – Uniform distribution on a unit disk



Suppose that a pair of random variabes (X, Y) is chosen uniformly from the set of real points (x, y) such that $x^2 + y^2 \leq 1$

This is a disk of radius 1 which has area π

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Poll: pollev.com/stefanotessaro617

Are X and Y independent?

a. Yes

b. No

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} \mathrm{d}y$$

$$= 2\sqrt{1-x^2}/\pi$$

Covariance: How correlated are X and Y?

Recall that if X and Y are independent, $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Definition: The **covariance** of random variables X and Y,

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Unlike variance, covariance can be positive or negative. It has has value 0 if the random variables are independent.

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Two Covariance examples:

Suppose $X \sim \text{Bernoulli}(p)$

If random variable $Y = X$ then

$$\text{Cov}(X, Y) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \text{Var}(X) = p(1-p)$$

If random variable $Z = -X$ then

$$\begin{aligned}\text{Cov}(X, Z) &= \mathbb{E}[XZ] - \mathbb{E}[X] \cdot \mathbb{E}[Z] \\ &= \mathbb{E}[-X^2] - \mathbb{E}[X] \cdot \mathbb{E}[-X] \\ &= -\mathbb{E}[X^2] + \mathbb{E}[X]^2 = -\text{Var}(X) = -p(1-p)\end{aligned}$$

$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Compare with …

Suppose $X, Y \sim \text{Bernoulli}(p)$, independent

Then: $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[X] \cdot \mathbb{E}[Y] = 0$

Joint Expectation

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs X and Y is

$$\mathbb{E}[g(X,Y)] = \sum_{a\in\Omega_X} \sum_{b\in\Omega_Y} g(a,b) \cdot p_{X,Y}(a,b)$$

Definition. Let X and Y be continuous random variables and $f_{X,Y}(x,y)$ their joint PDF. The **expectation** of some function $g(x,y)$ with inputs X and Y is

$$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)\, \mathrm{d}x\, \mathrm{d}y$$

Brain Break

Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
- Conditional Expectation and Law of Total Expectation

Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event A is

$$\mathbb{E}[X \mid A] = \sum_{x \in \Omega_X} x \cdot P(X = x \mid A)$$

Notes:

- Can be phrased as a "random variable version"

$$\mathbb{E}[X \mid Y = y]$$

- Linearity of expectation still applies here

$$\mathbb{E}[aX + bY + c \mid A] = a\,\mathbb{E}[X \mid A] + b\,\mathbb{E}[Y \mid A] + c$$

Law of Total Expectation

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \ldots, A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \cdot P(A_i)$$

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X \mid Y = y] \cdot P(Y = y)$$

Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

$$= \sum_{x \in \Omega_X} x \cdot \sum_{i=1}^{n} P(X = x \mid A_i) \cdot P(A_i)$$ (by LTP)

$$= \sum_{i=1}^{n} P(A_i) \sum_{x \in \Omega_X} x \cdot P(X = x \mid A_i)$$ (change order of sums)

$$= \sum_{i=1}^{n} P(A_i) \cdot \mathbb{E}[X \mid A_i]$$ (def of cond. expect.)

Example – Flipping a Random Number of Coins

Suppose someone gave us $Y \sim \mathrm{Poi}(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.

By the Law of Total Expectation

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} \mathbb{E}[X \mid Y = i] \cdot P(Y = i) = \sum_{i=0}^{\infty} \frac{i}{2} \cdot P(Y = i)$$

$$= \frac{1}{2} \cdot \sum_{i=0}^{\infty} i \cdot P(Y = i)$$

$$= \frac{1}{2} \cdot \mathbb{E}[Y] = \frac{1}{2} \cdot 5 = 2.5$$

Example – Computer Failures (a familiar example)

Suppose your computer operates in a sequence of steps, and that at each step i your computer will fail with probability p (independently of other steps).

Let X be the number of steps it takes your computer to fail.

What is $\mathbb{E}[X]$?

Let Y be the indicator random variable for the event of failure in step 1

Then by LTE, $\mathbb{E}[X] = \mathbb{E}[X \mid Y = 1] \cdot P(Y = 1) + \mathbb{E}[X \mid Y = 0] \cdot P(Y = 0)$

$$= 1 \cdot p + \mathbb{E}[X \mid Y = 0] \cdot (1 - p)$$

$$= p + (1 + \mathbb{E}[X]) \cdot (1 - p)$$

since if $Y = 0$ experiment starting at step 2 looks like original experiment

Solving we get $\mathbb{E}[X] = 1/p$

Reference Sheet (with continuous RVs)

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leq x} \sum_{s \leq y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X\mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X\mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$