CSE 312

Foundations of Computing II

Lecture 16: CLT & Polling

Review CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal = $\mathcal{N}(0, 1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

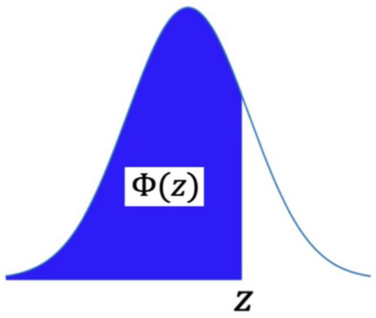
Note: $\Phi(z)$ has no closed form – generally given via tables

Review

Table of $\Phi(z)$ CDF of Standard Normal Distribution

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999



Review Analyzing non-standard normal in terms of $\mathcal{N}(0,1)$

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$

Therefore,

$$F_X(z) = P(X \le z) = P\left(\frac{X-\mu}{\sigma} \le \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

Review How Many Standard Deviations Away?

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$P(|X-\mu| < k\sigma) = P\left(\frac{|X-\mu|}{\sigma} < k\right) =$$

$$= P\left(-k < \frac{X-\mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%
$k = 2$: 95%
$k = 3$: 99%

Review Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\left(\mathrm{Var}(S_n - n\mu)\right) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Review Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Also stated as:

- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}(X_i)$

Consequence of Central Limit Theorem:

$S_n = X_1 + \cdots + X_n$

gets more concentrated around mean as n gets large!

What is the probability that S_n deviates from mean by 1% ?

$$\mathrm{P}(|\mathrm{S_n} - \mathrm{n}\,\mu| \geq \frac{n\mu}{100}) \;=\; P(\left|\frac{S_n - n\mu}{\sigma\sqrt{n}}\right| \geq \frac{\sqrt{n}\mu}{100\sigma})$$

$\approx$ probability that standard normal deviates from 0 by $\frac{\sqrt{n}\mu}{100\,\sigma}$ standard deviations. Goes to 0 as soon as $\frac{\sqrt{n}\mu}{100\,\sigma} \gg 1.$

Agenda

- Central Limit Theorem (CLT) Review
- Polling

Predicting the outcome of elections

Poll to determine the fraction p of the population expected to vote for Harris or Trump.

- Call up a random sample of n people to ask their opinion
- Report the empirical fraction

Questions

- Is this a good estimate?
- How to choose n?

Polling Accuracy

Often see claims that say

> *"Our poll found 80% support. This poll is accurate to within 5% with 98% probability"*

Will unpack what this and how they sample enough people to know this is true.

Formalizing Polls

Population size N, true fraction of voting in favor p, sample size n.

Problem: We don't know p

Polling Procedure

for $i = 1, \ldots, n$:

1. Pick uniformly random person to call (prob: $1/N$)
2. Ask them how they will vote

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

Report our estimate of p: $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Random Variables

Population size N, true fraction of voting in favor p, sample size n.

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

What are the statistics of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

	$\mathbb{E}[\bar{X}]$	$\mathrm{Var}(\bar{X})$
a.	np	$np(1-p)$
b.	p	$p(1-p)$
c.	p	$p(1-p)/n$
d.	p/n	$p(1-p)/n$

Random Variables

Population size N, true fraction of voting in favor p, sample size n.

$$X_i = \begin{cases} 1, & \text{voting in favor} \\ 0, & \text{otherwise} \end{cases}$$

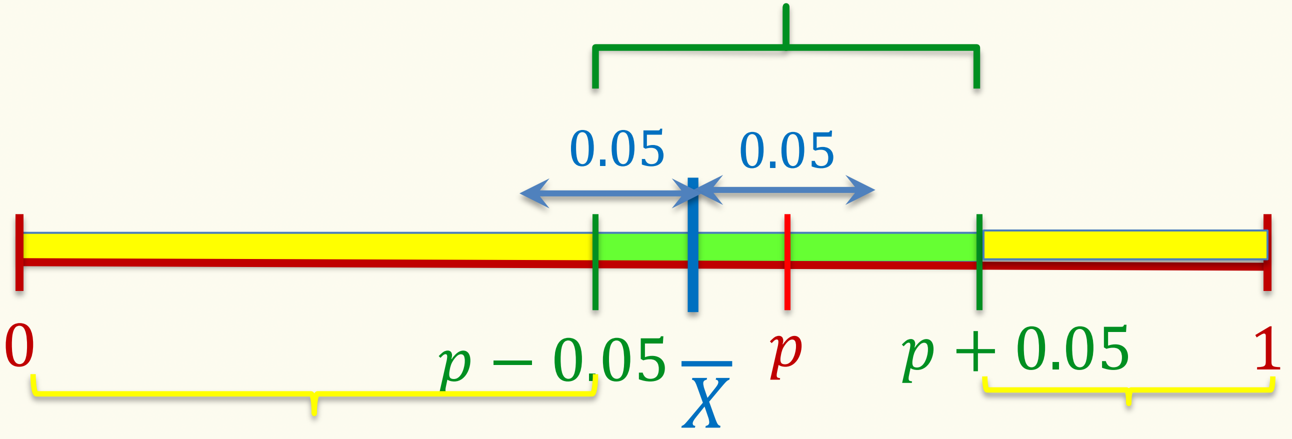
What are the statistics of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$?

$\mathbb{E}[\bar{X}]$	$\mathrm{Var}(\bar{X})$
p	$p(1-p)/n$

Roadmap: Bounding Error

Goal: Find the value of n such that 98% of the time, the estimate $\overline{X}$ is within 5% of the true p

Get good estimate if $\overline{X}$ lands in this region



Want $P\left(\left|\overline{X} - p\right| > 0.05\right) \leq 0.02$

Central Limit Theorem

In the limit $\overline{X}$ is...
$\mathcal{N}(p, p(1-p)/n)$

With i.i.d random variables $X_1, X_2, \ldots, X_n$ where $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$

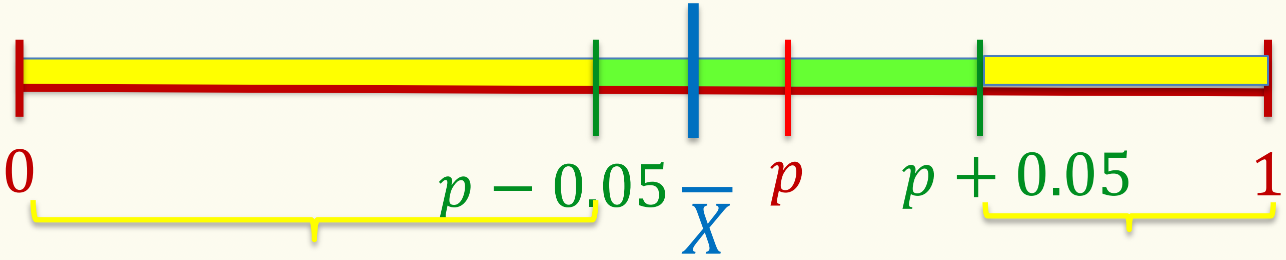
As $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots X_n \; - n\mu}{\sigma\sqrt{n}} \to \mathcal{N}(0,1)$$

Restated: As $n \to \infty$,

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

Roadmap: Bounding Error



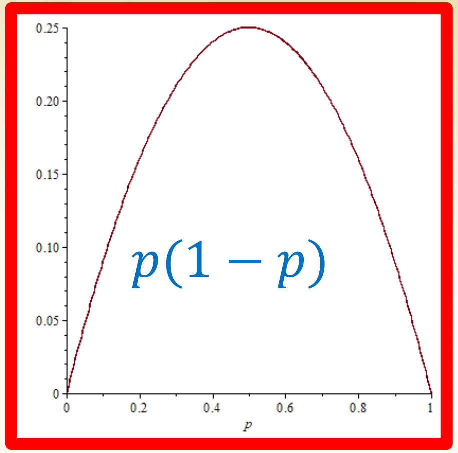
Want $P\left(\left|\overline{X} - p\right| > 0.05\right) \leq 0.02$

Roadmap: Bounding Error

Goal: Find the value of n such that 98% of the time, the estimate $\bar{X}$ is within 5% of the true p

1. Define probability of a "bad event" $P\left(\left|\bar{X} - p\right| > 0.05\right) \le 0.02$
2. Apply CLT
3. Convert to a standard normal
4. Solve for n

Following the Road Map



1. Want $P\left(\left|\overline{X}-p\right|>0.05\right)\le 0.02$
2. By CLT $\overline{X}\to\mathcal{N}(\mu,\sigma^2)$ where $\mu=p$ and $\sigma^2=p(1-p)/n$
3. Define $Z=\frac{\overline{X}-\mu}{\sigma}=\frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z\to\mathcal{N}(0,1)$

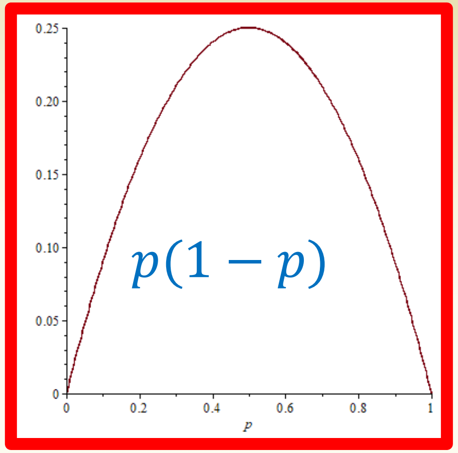
$\frac{\mathbf{1}}{\sqrt{\boldsymbol{p}(\mathbf{1}-\boldsymbol{p})}}$ is always $\ge \mathbf{2}$

$$P\left(\left|\overline{X}-p\right|>0.05\right)=P(|Z|\cdot\sigma>0.05)$$

$$=P(|Z|>0.05/\sigma)=P\left(|Z|>0.05\frac{\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

$$\le P(|Z|>0.1\sqrt{n})$$

Following the Road Map



1. Want $P\left(\left|\overline{X}-p\right| > 0.05\right) \le 0.02$
2. By CLT $\overline{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = p$ and $\sigma^2 = p(1-p)/n$
3. Define $Z = \frac{\overline{X}-\mu}{\sigma} = \frac{\overline{X}-p}{\sigma}$. Then, by the CLT $Z \to \mathcal{N}(0,1)$

$\boldsymbol{\frac{1}{\sqrt{p(1-p)}}}$ is always $\ge \mathbf{2}$

$$P\left(\left|\overline{X}-p\right| > 0.05\right) = P(|Z| \cdot \sigma > 0.05)$$

$$= P(|Z| > 0.05/\sigma) = P\left(|Z| > 0.05\frac{\sqrt{n}}{\sqrt{p(1-p)}}\right)$$

$$\le P(|Z| > 0.1\sqrt{n})$$

Want to choose n so that this is at most 0.02

4. Solve for n

We want $P(|Z| > 0.1\sqrt{n}) \leq 0.02$ *where* $Z \to \mathcal{N}(0,1)$

- If we actually had $Z \sim \mathcal{N}(0,1)$ then enough to show that $P(Z > 0.1\sqrt{n}) \leq 0.01$ since $\mathcal{N}(0,1)$ is symmetric about 0
- Now $P(Z > z) = 1 - \Phi(z)$ where $\Phi(z)$ is the CDF of the Standard Normal Distribution
- So, want to choose n so that $0.1\sqrt{n} \geq z$ where $\Phi(z) \geq 0.99$

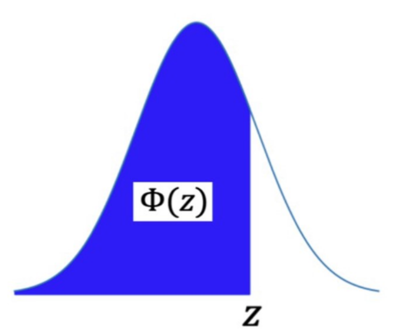
Table of $\Phi(z)$ CDF of Standard Normal Distribution

Choose n so

$0.1\sqrt{n} \geq z$ where

$\Phi(z) \geq 0.99$

From table $z = 2.33$ works



Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

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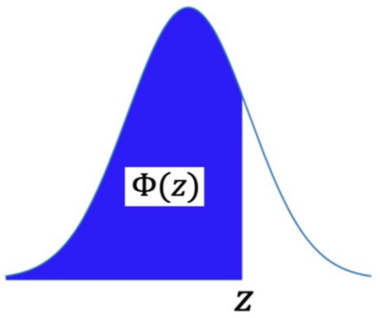
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

4. Solve for n

Choose n so
$0.1\sqrt{n} \geq z$ where
$\Phi(z) \geq 0.99$

From table $z = 2.33$ works



- So we can choose $0.1\sqrt{n} \geq 2.33$
  or $\sqrt{n} \geq 23.3$
- Then $n \geq 543 \geq (23.3)^2$ would be good enough … if we had $Z \sim \mathcal{N}(0, 1)$
- We only have $Z \to \mathcal{N}(0, 1)$ so there is some loss due to approximation error.
- Maybe instead consider $z = 3.0$ with $\Phi(z) \geq 0.99865$ and $n \geq 30^2 = 900$ to cover any loss.

Idealized Polling

So far, we have been discussing "idealized polling". Real life is normally not so nice ☹

Assumed we can sample people uniformly at random, not really possible in practice

- Not everyone responds
- **Response rates might differ in different groups**
- Will people respond truthfully?

Makes polling in real life much more complex than this idealized model!

The problem

pN will vote Harris, $(1-p)N$ will vote Trump.

h fraction Harris voters answer phone.

t fraction Trump voters answer phone.

$\overline{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = \frac{ph}{ph+t(1-p)}$ and $\sigma^2 = \mu(1-\mu)/n$.

The problem

pN will vote Harris, $(1-p)N$ will vote Trump.

h fraction Harris voters answer phone.

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$\overline{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = \frac{ph}{ph+t(1-p)}$ and $\sigma^2 = \mu(1-\mu)/n$.

If $\mathrm{h} = t$, then $\mu = p$! But in general $h \neq t$!

The solution??

pN will vote Harris, $(1 - p)N$ will vote Trump.

h fraction Harris voters answer phone.

t fraction Trump voters answer phone.

$\overline{X} \to \mathcal{N}(\mu, \sigma^2)$ where $\mu = \frac{ph}{ph+t(1-p)}$ and $\sigma^2 = \mu(1 - \mu)/n$.

Maybe: Within a certain demographic (e.g. women of age 25-30), $\mathrm{h} = t$. Then we can estimate p for that demographic, and if we can do this for all demographics, we can estimate overall p.