CSE 312

Foundations of Computing II

Lecture 15: Normal Distribution & Central Limit Theorem

Review Continuous RVs

Probability Density Function (PDF).

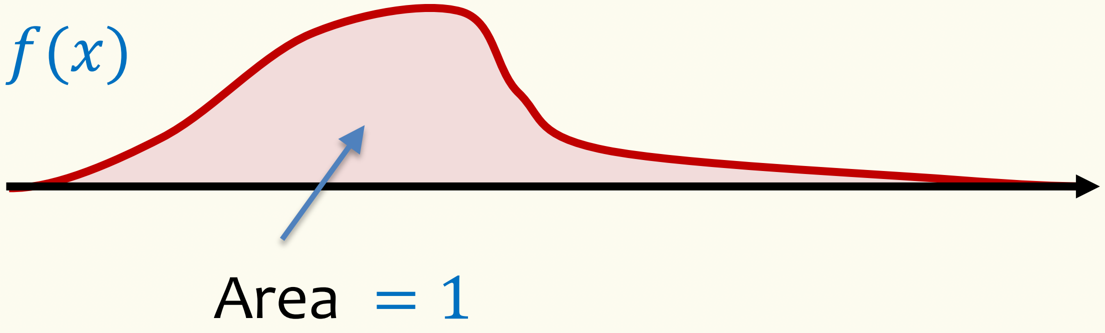
$f: \mathbb{R} \to \mathbb{R}$ s.t.

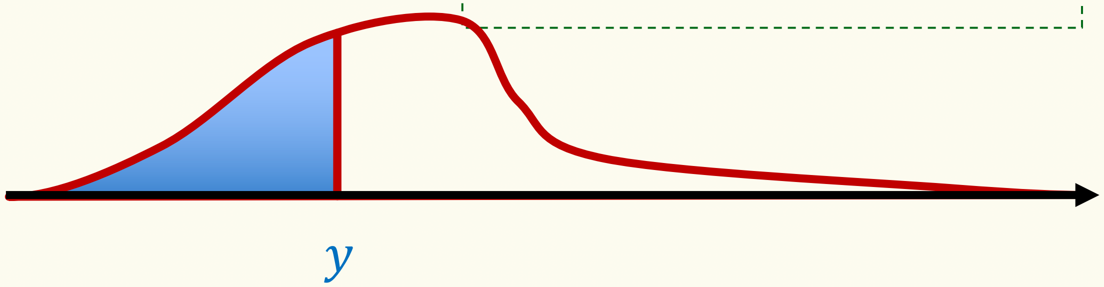
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$

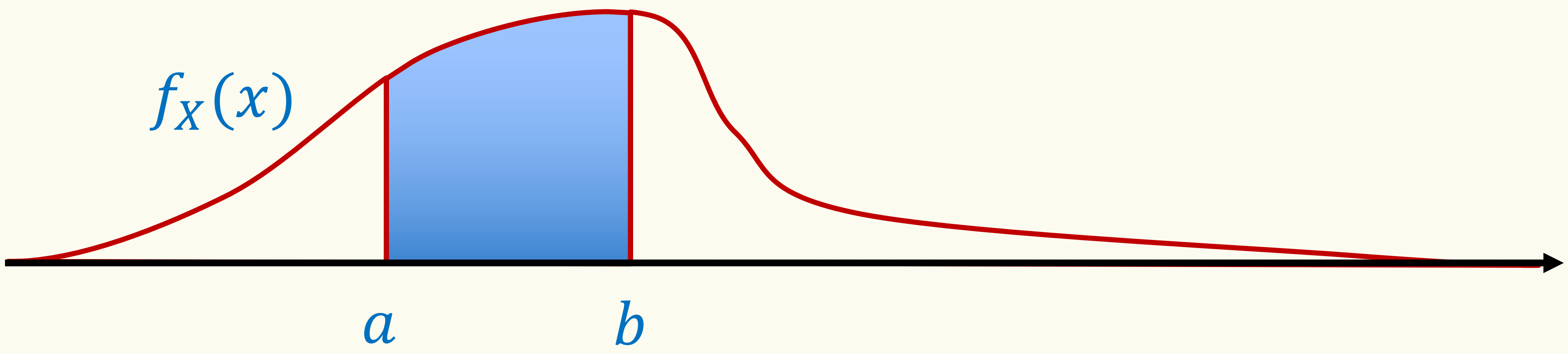




Density $\neq$ Probability !

$F_X(y) = P(X \leq y)$

Review Continuous RVs



$$P(X \in [a,b]) = \int_a^b f_X(x)\mathrm{d}x = F_X(b) - F_X(a)$$

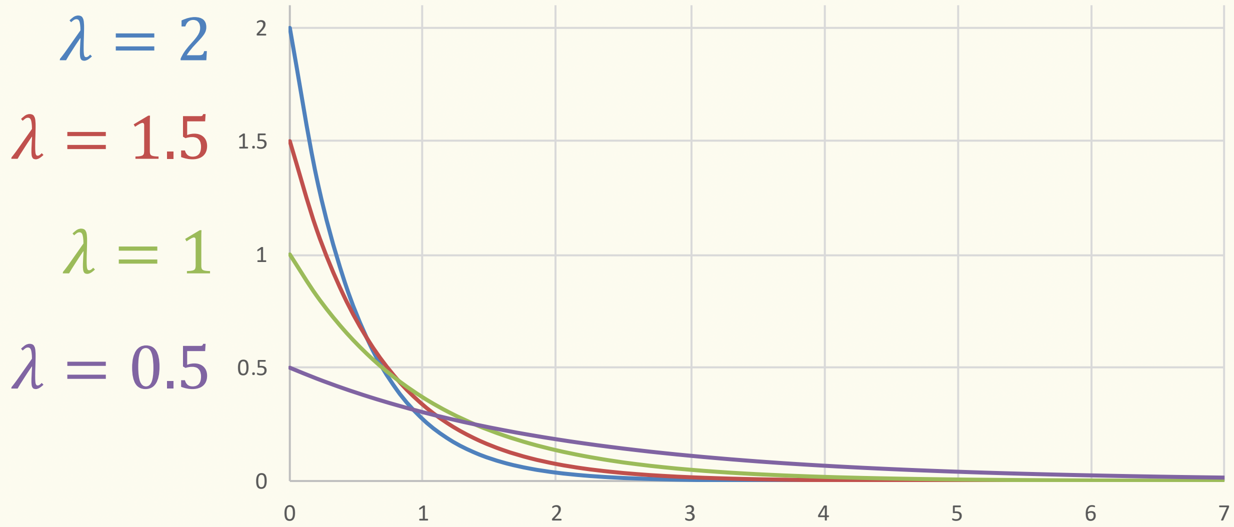
Review Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$$F_X(y) = 1 - e^{-\lambda y}$$



Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT)

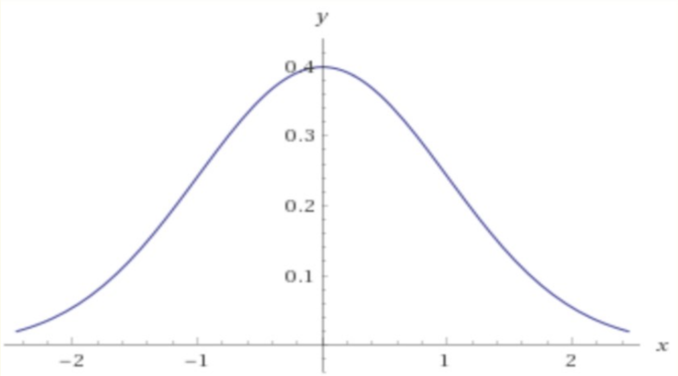
The Normal Distribution

Carl Friedrich Gauss

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.



$\mathcal{N}(0, 1)$.

The Normal Distribution

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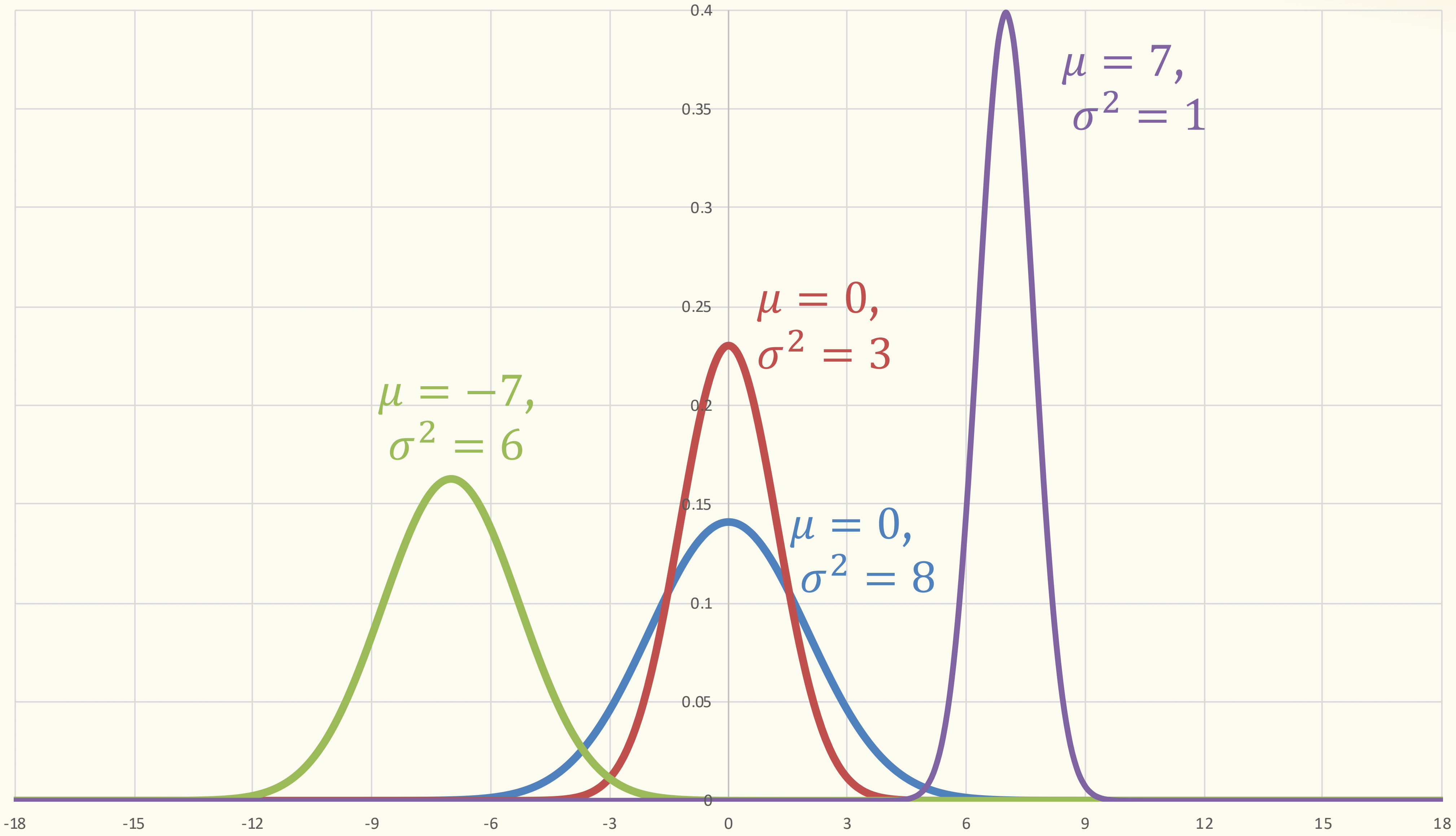
Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around μ,
$f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

The Normal Distribution

Aka a “Bell Curve” (imprecise name)



Closure of normal distribution – Under Shifting and Scaling

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Proof.

$$\mathbb{E}[Y] = a\,\mathbb{E}[X] + b = a\mu + b$$

$$\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X) = a^2\sigma^2$$

Can show with algebra that the PDF of $Y = aX + b$ is still normal.

Note: $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

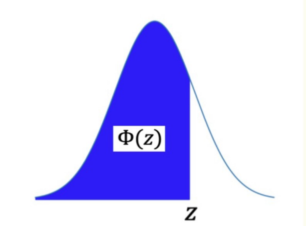
Standard (unit) normal = $\mathcal{N}(0, 1)$

CDF. $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \mathrm{d}x$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no nice formula!

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi(\frac{z-\mu}{\sigma})$

Table of Standard Cumulative Normal Density



$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$

What is
$P(Z \leq -1.09)$?

Φ Table: $\mathbb{P}(Z \leq z)$ when $Z \sim \mathcal{N}(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

Closure of the normal -- under addition

Fact. If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $\mathrm{Y} \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $\mathrm{a}X + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that **the sum of normal RVs is still a normal RV.**
The values of the expectation and variance are **not** surprising.

Why not surprising?

- Linearity of expectation (always true)
- When X and Y are independent, $\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y)$

Agenda

- Normal Distribution
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- Central Limit Theorem (CLT)

What about Non-standard normal?

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$$

Example

Let $X \sim \mathcal{N}(0.4, 4 = 2^2)$.

$$P(X \leq 1.2) = P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right)$$

$$= P\left(\underbrace{\frac{X - 0.4}{2}}_{\sim \mathcal{N}(0,1)} \leq 0.4\right) = \Phi(0.4) \approx 0.6554$$

Example

Let $X \sim \mathcal{N}(3, 16)$.

$$P(2 < X < 5) = P\left(\frac{2 \ - 3}{4} < \frac{X - 3}{4} < \frac{5 \ - 3}{4}\right)$$

$$= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right)$$

$$= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017$$

Example – How Many Standard Deviations Away?

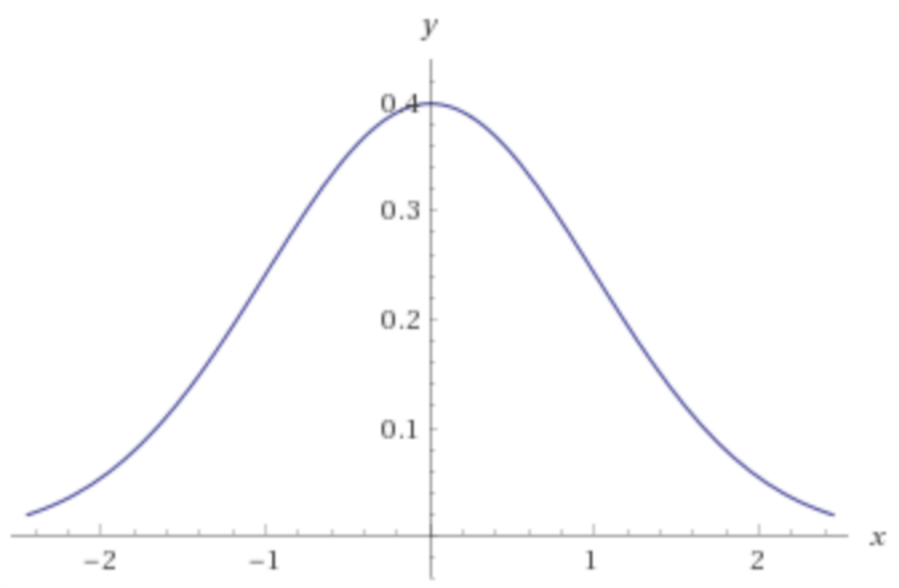
Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$P(|X-\mu| < k\sigma) = P\left(\frac{|X-\mu|}{\sigma} < k\right) =$$

$$= P\left(-k < \frac{X-\mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%
$k = 2$: 95%
$k = 3$: 99%

Halloween Brain Break



Normal Distribution

Paranormal Distribution

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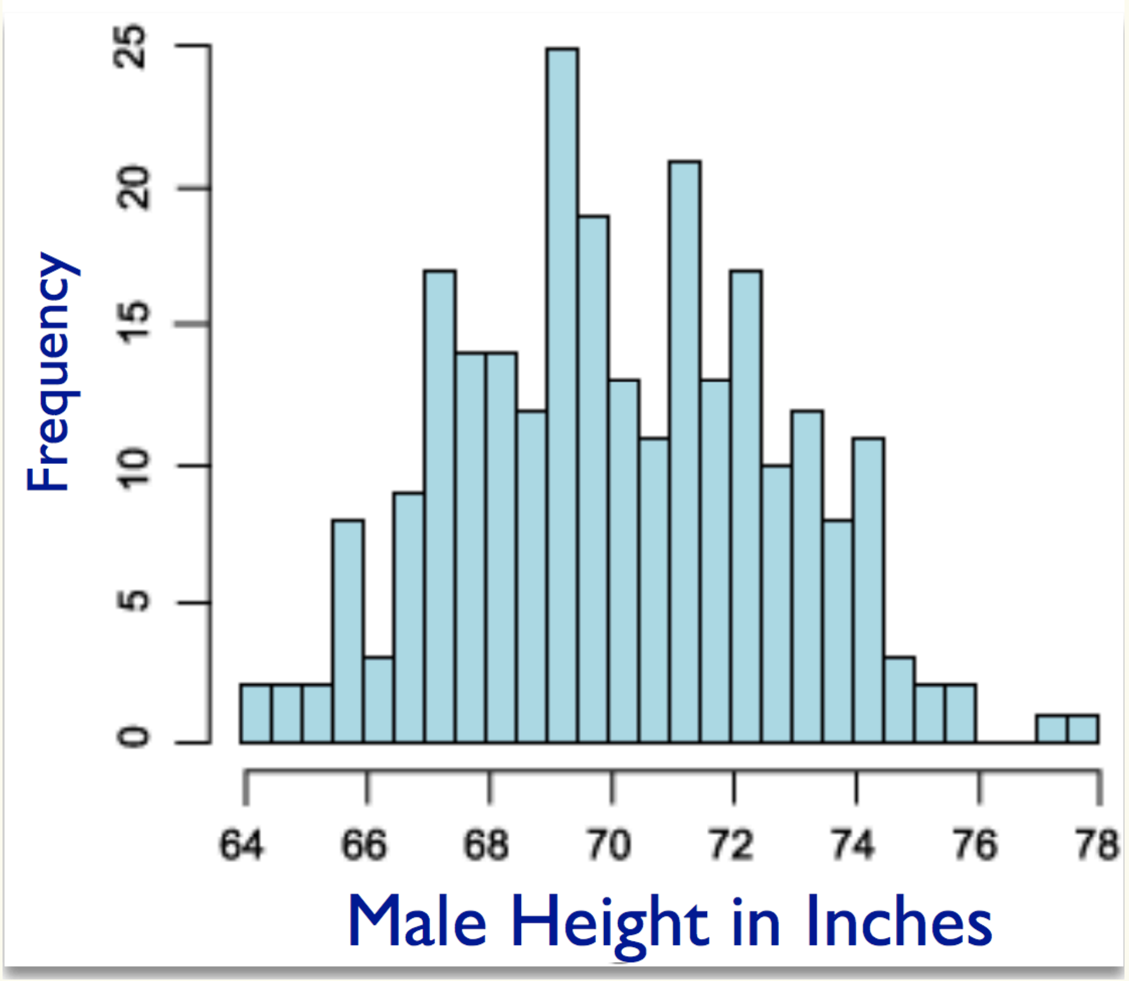
Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian …



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

$$X = X_1 + \cdots + X_n$$

Sum of Independent RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$ i.i.d. with expectation μ and variance σ^2

Define

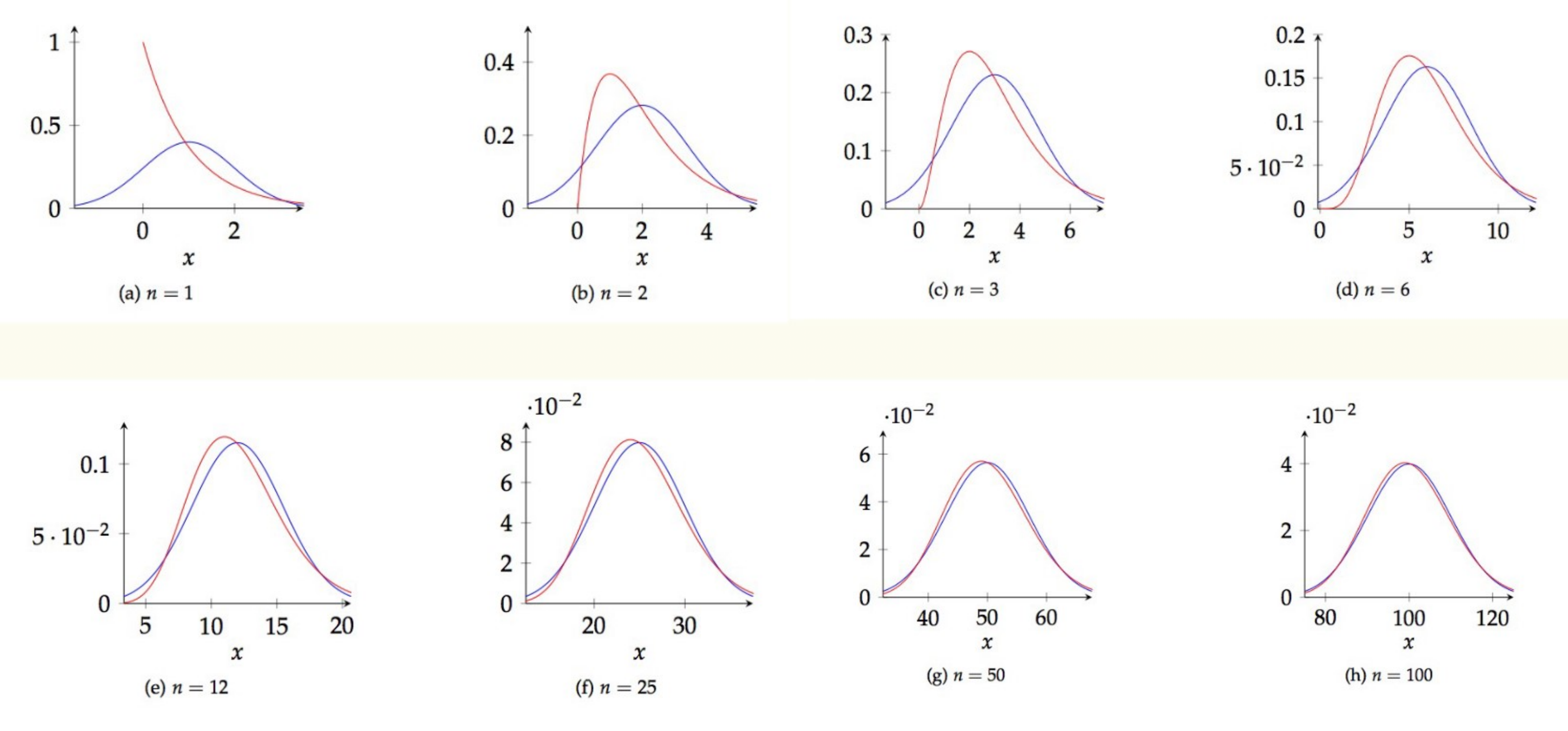
$$S_n = X_1 + \cdots + X_n$$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

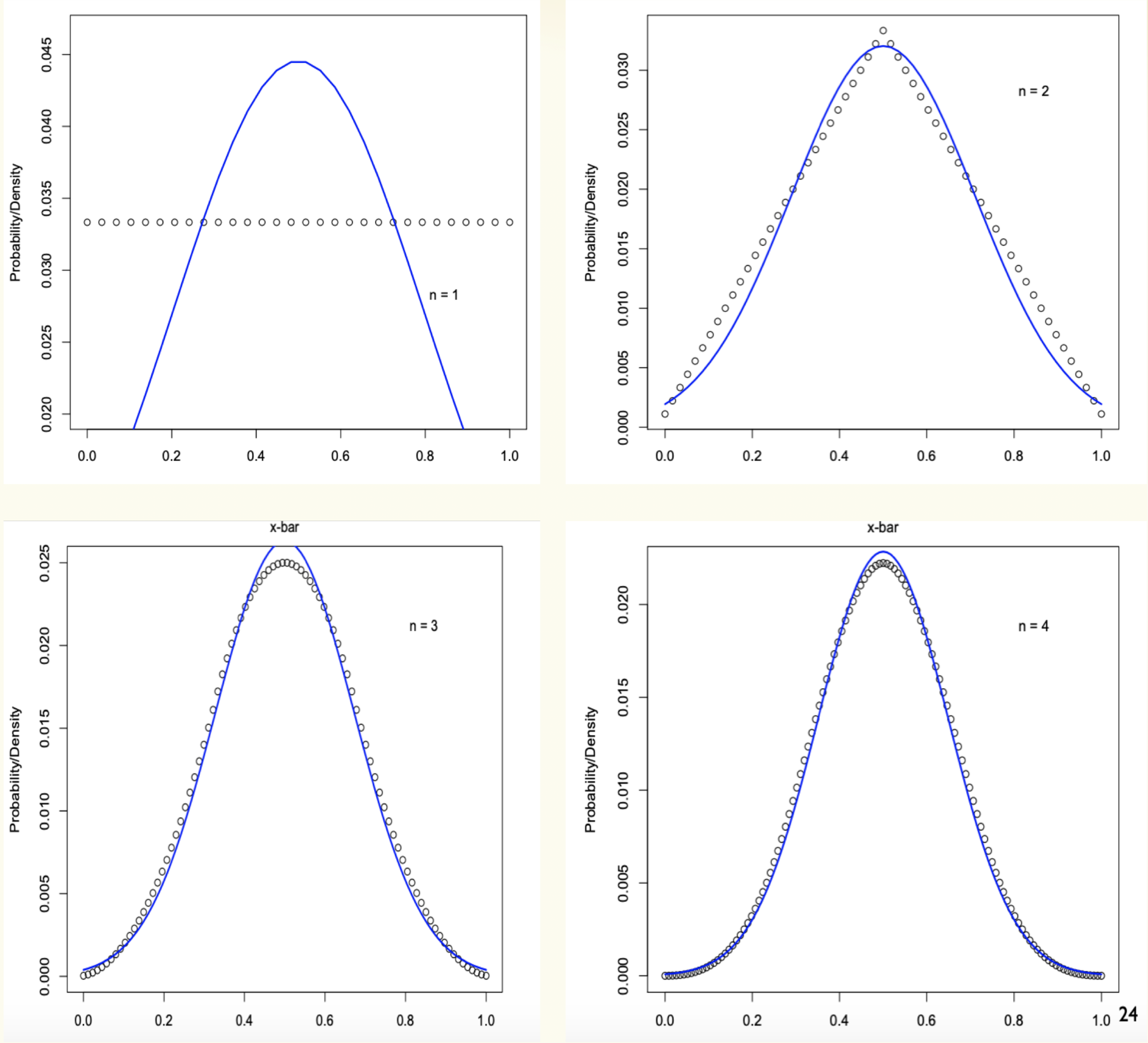
$$\mathrm{Var}(S_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) = n\sigma^2$$

Empirical observation: S_n looks like a normal RV as n grows.

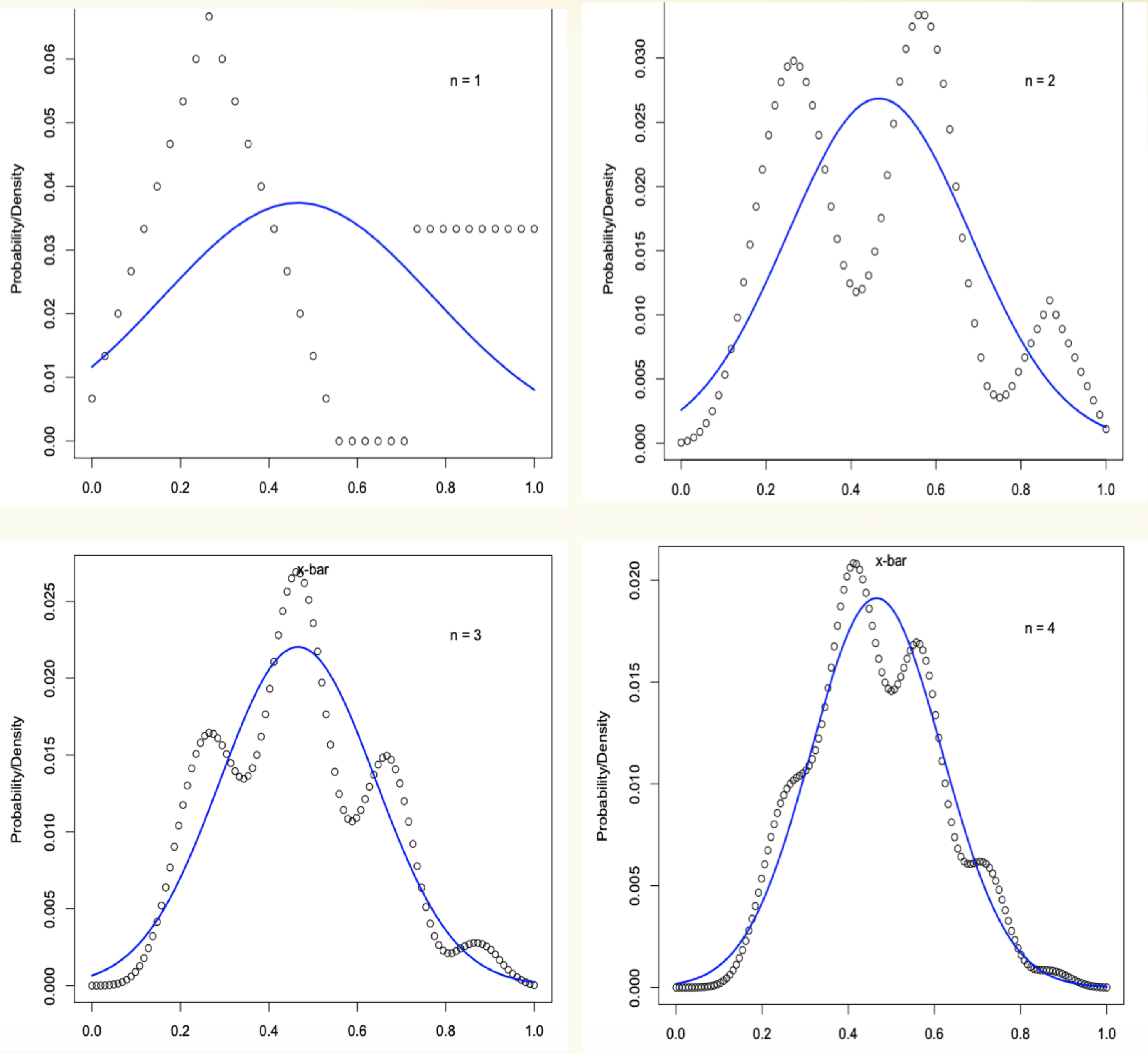
Example: Sum of n i.i.d. $\mathrm{Exp}(1)$ random variables



CLT (Idea)



CLT (Idea)



n = 1
n = 2
n = 3
n = 4
x-bar
x-bar
Probability/Density
Probability/Density
Probability/Density
Probability/Density
0.0
0.2
0.4
0.6
0.8
1.0

Central Limit Theorem

$X_1, \dots, X_n$ i.i.d., each with expectation μ and variance σ^2

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}}(\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}}(n\mu - n\mu) = 0$$

$$\mathrm{Var}(Y_n) = \frac{1}{\sigma^2 n}\left(\mathrm{Var}(S_n - n\mu)\right) = \frac{\mathrm{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

Theorem. (Central Limit Theorem) The CDF of Y_n converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$\lim_{n\to\infty} P(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

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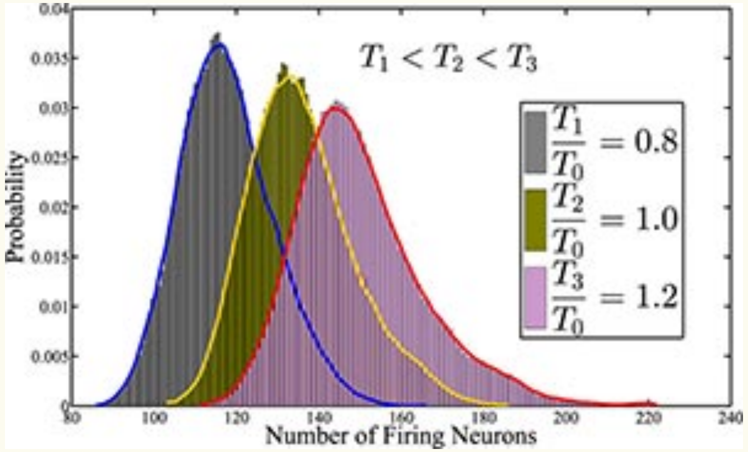
$$\lim_{n\to\infty} P(Y_n \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \mathrm{d}x$$

Also stated as:

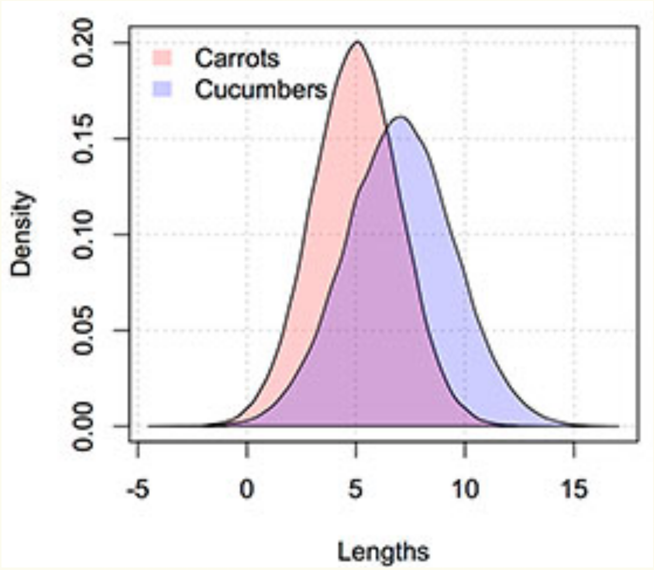
- $\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$
- $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$ for $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}(X_i)$

CLT → Normal Distribution EVERYWHERE

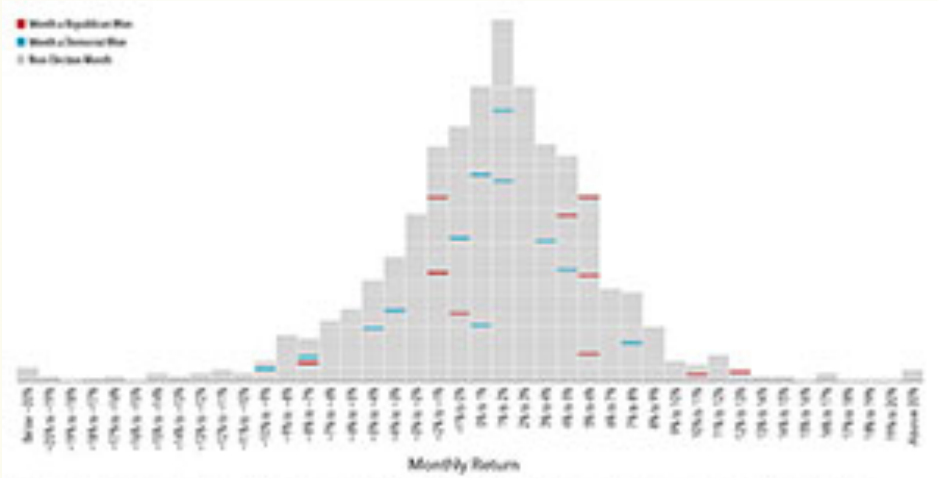
Neuron Activity



Vegetables



S&P 500 Returns after Elections



Examples from:
https://galtonboard.com/probabilityexamplesinlife