CSE 312

Foundations of Computing II

Lecture 13: Poisson wrap-up
Continuous RV

Announcements

- PSet 4 due today
- Midterm next week in class!
- Midterm general info is posted on Ed
- Review session likely on Tuesday night
- Practice midterm is posted

Agenda

- Wrap-up of Poisson RVs
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Disjoint time intervals independent
- Number of events happening at a time unit X is distributed according to Poi(λ)

- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Sum of Independent Poisson RVs

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \cdots, X_n \sim \text{Poi}(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.

Let $Z = \Sigma_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Proof

$Z = X + Y$ where $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ are independent

$$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$$ Law of total probability

$$= \Sigma_{j=0}^{z} P(X = j)\, P(Y = z - j) = \Sigma_{j=0}^{z}\ e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z - j!}$$ Independence

$$= e^{-\lambda_1 - \lambda_2} \left(\Sigma_{j=0}^{z} \cdot \frac{1}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\Sigma_{j=0}^{z} \frac{z!}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

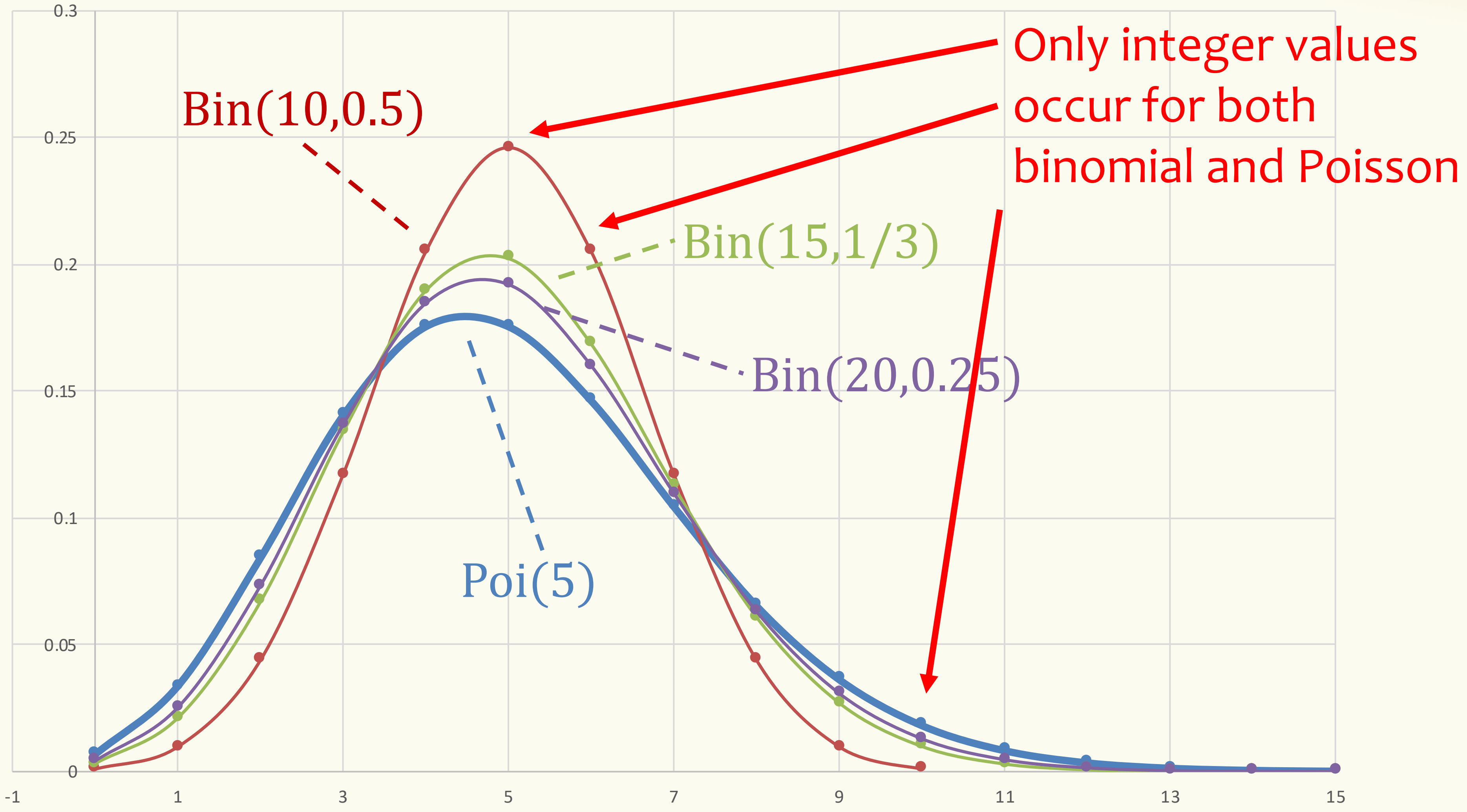
$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$ Binomial Theorem

Don’t be fooled by this picture: Poisson RVs are discrete

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10,15,20$



$as\ n \to \infty,\ \ Binomial(n,\ \ p = \ \lambda/n) \to poi(\lambda)$

Agenda

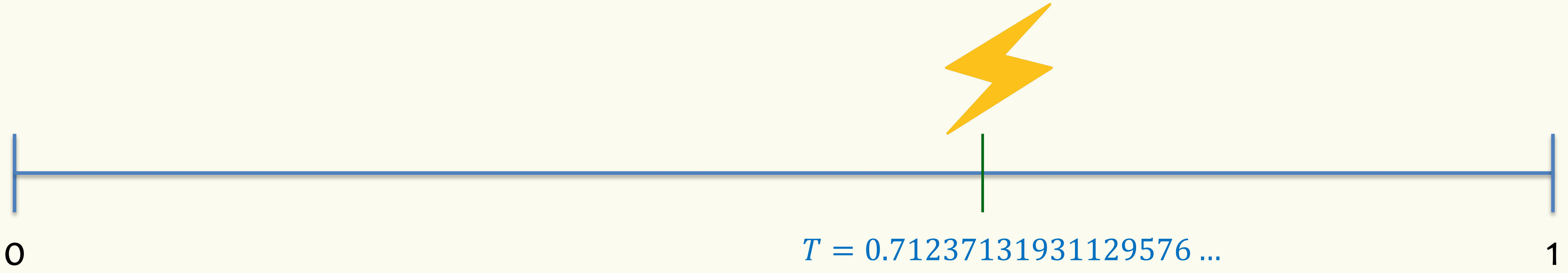
- Wrap-up of Poisson RVs
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function

Often we want to model experiments where the outcome is not discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  – Time measured with infinitesimal precision.

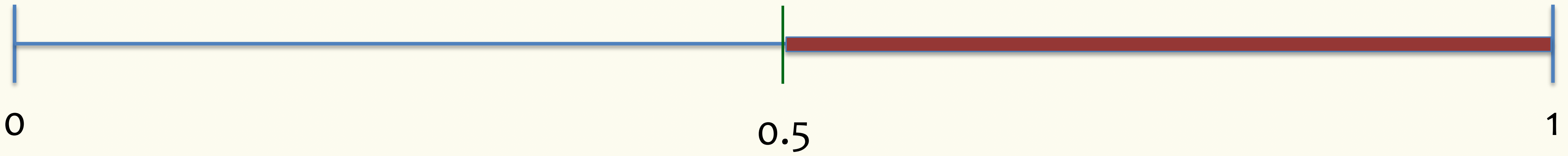


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

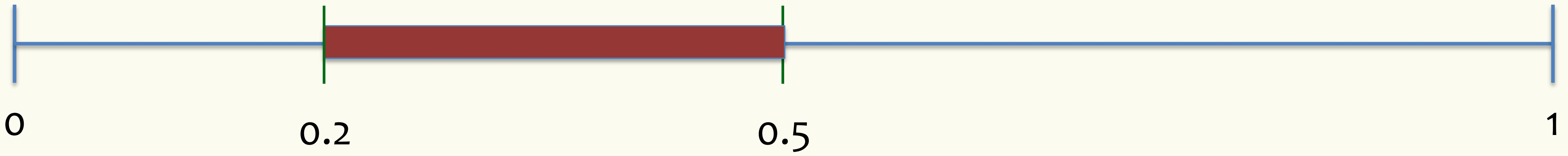
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T \geq 0.5) = ½$$



Lightning strikes a pole within a one-minute time frame

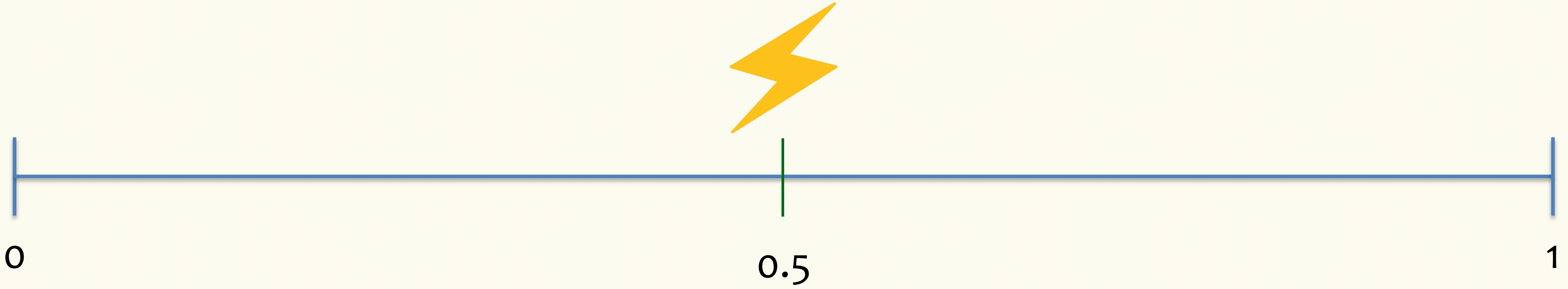
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



$$P(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3$$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



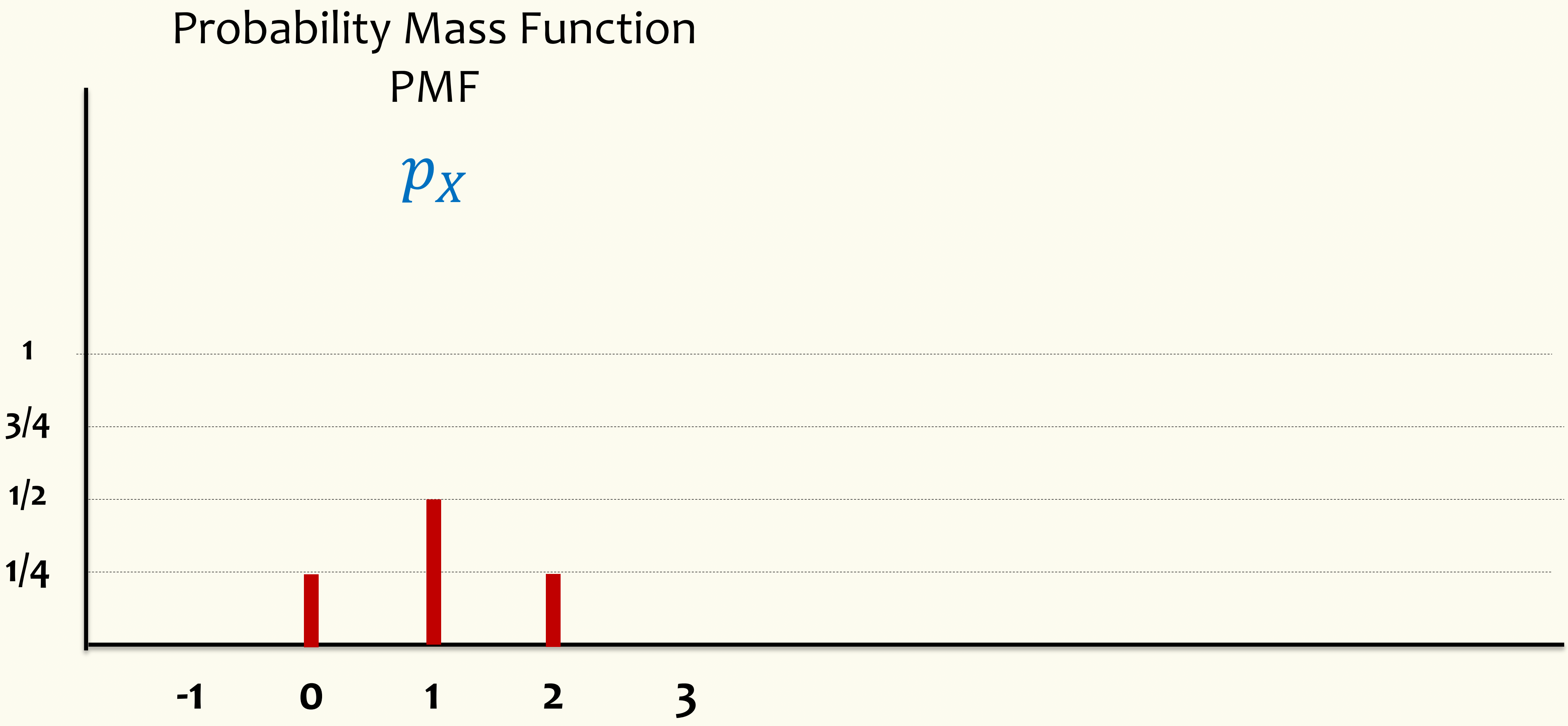
$$P(T = 0.5) = 0$$

Bottom line

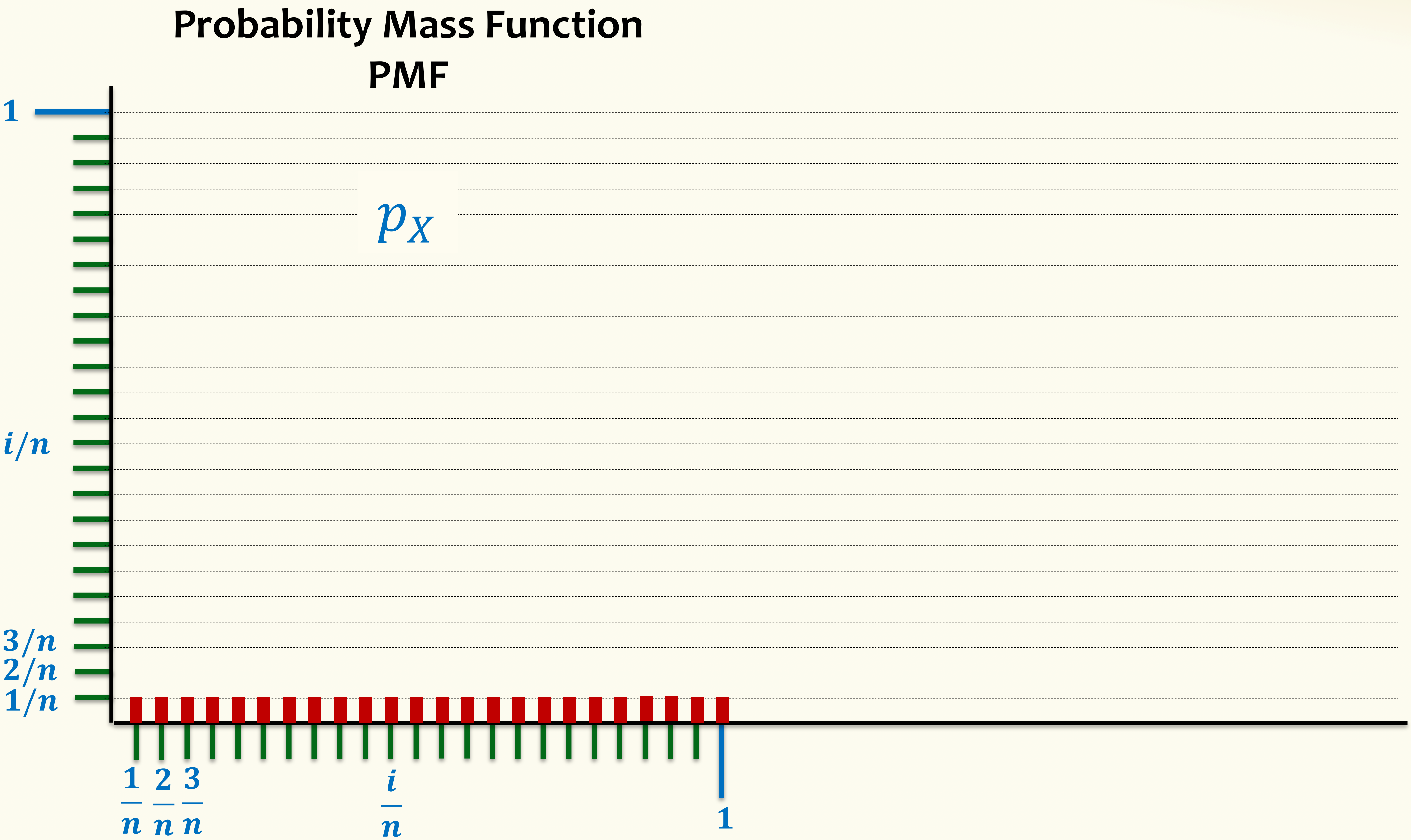
- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $P(T \in [0,1]) = 1$
  - $P(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of T?

First try: A discrete approximation

Recall: Cumulative Distribution Function (CDF)



A Discrete Approximation



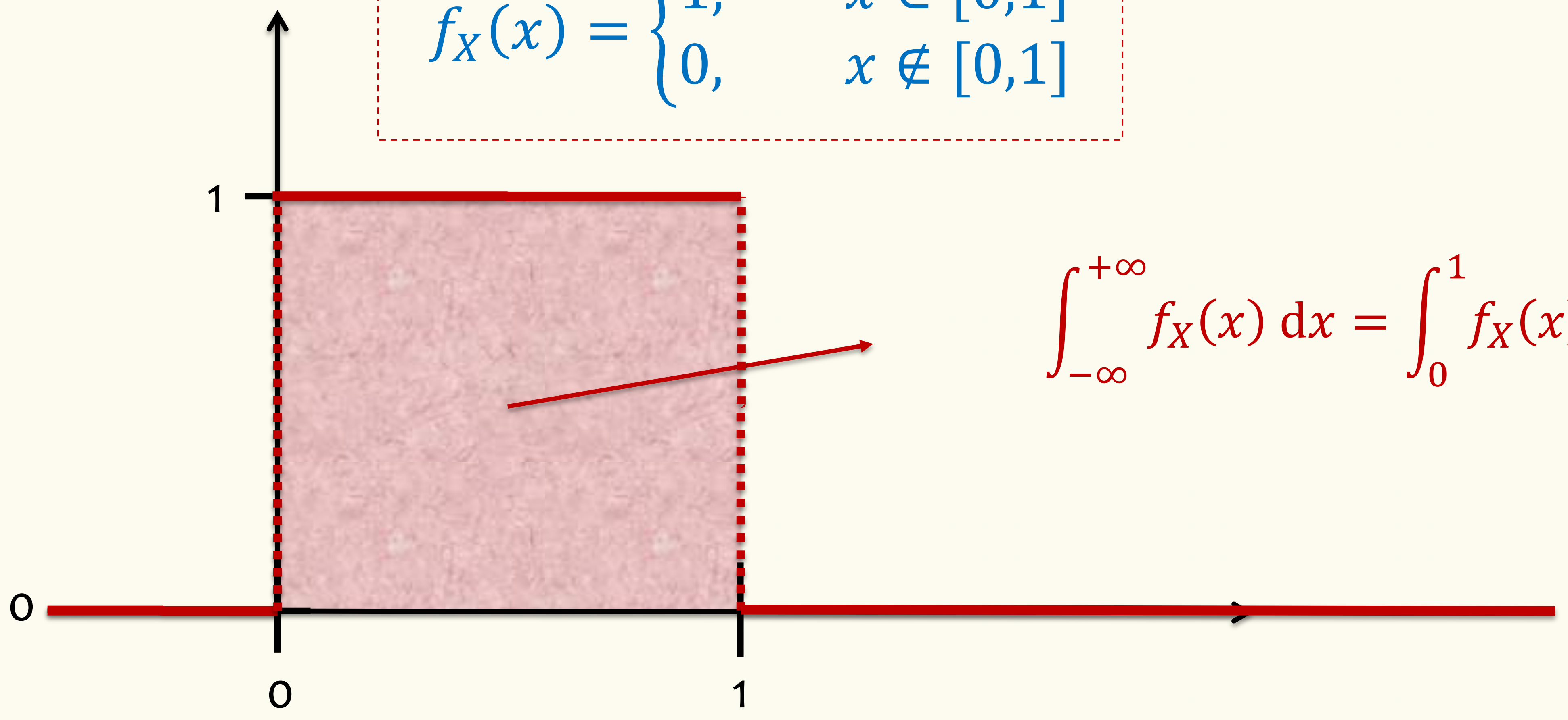
Probability Density Function of Uniform RV

$X \sim \text{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event

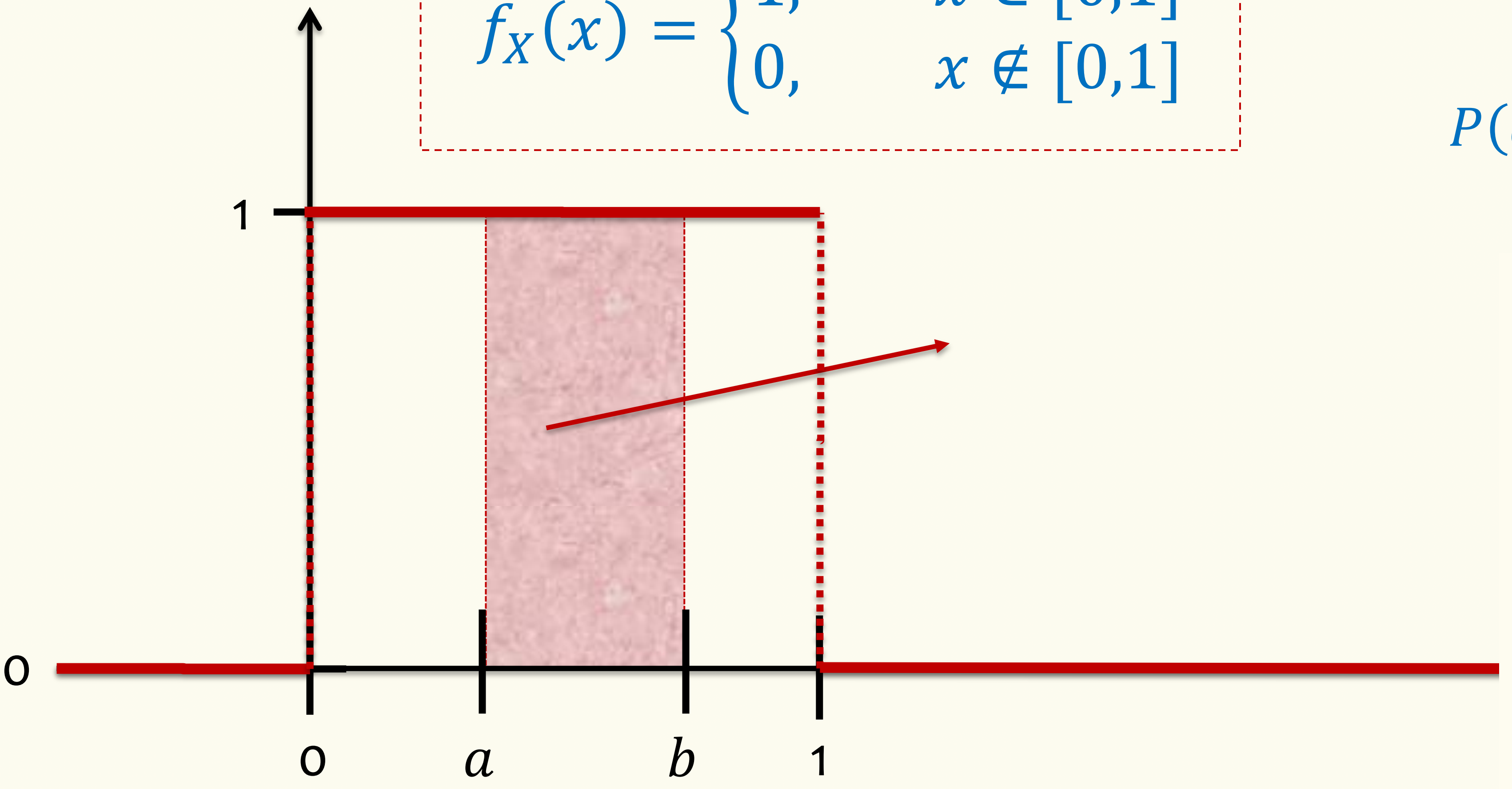
$X \sim \text{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

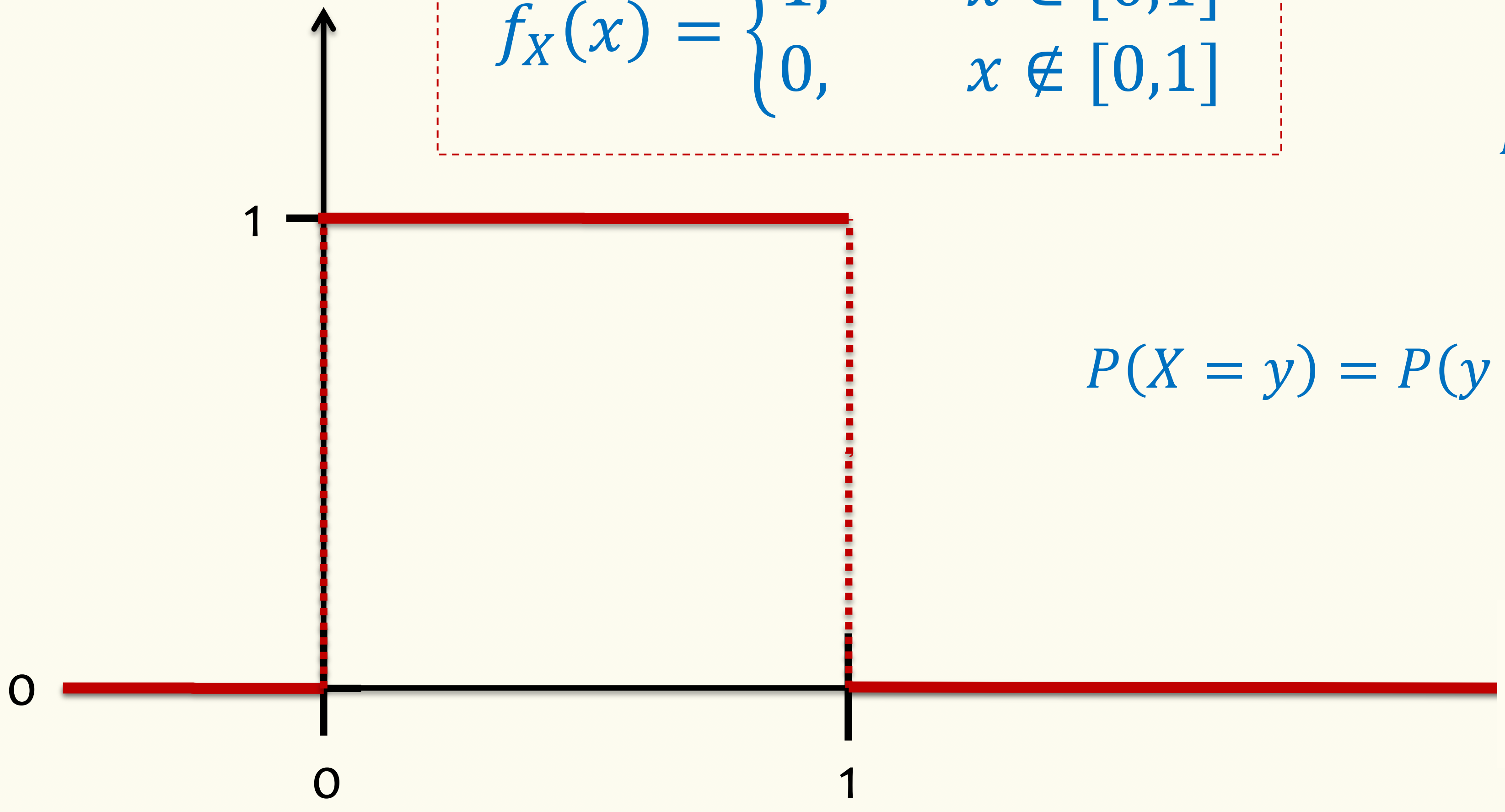


1. If $0 \leq a$ and $a \leq b \leq 1$
   $P(a \leq X \leq b) = b - a$
2. If $a < 0$ and $0 \leq b \leq 1$
   $P(a \leq X \leq b) = b$
3. If $a \geq 0$ and $b > 1$
   $P(a \leq X \leq b) = b - a$
4. If $a < 0$ and $b > 1$
   $P(a \leq X \leq b) = 1$

Probability of Event

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx \epsilon f_X(y) = \epsilon$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

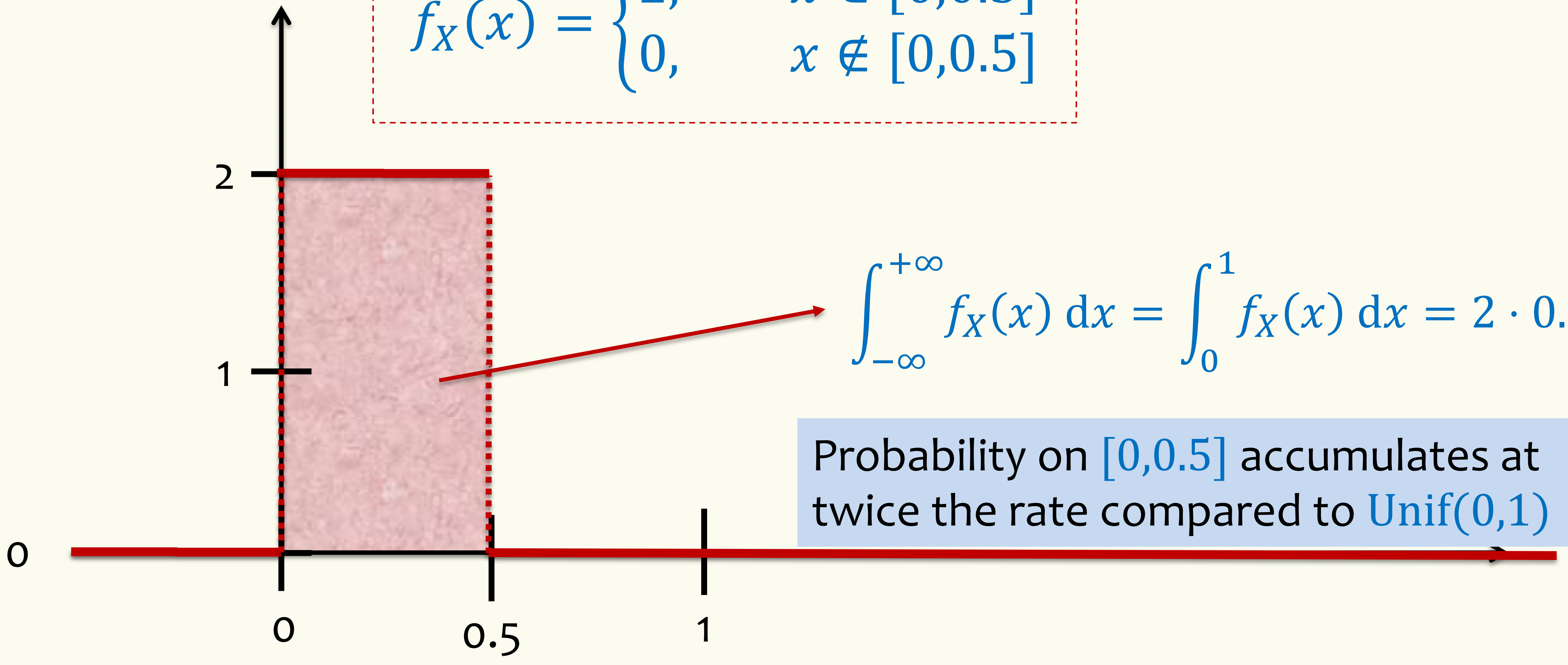
PDF of Uniform RV

Density ≠ Probability

$f_X(x) \gg 1$ is possible!

$X \sim \text{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



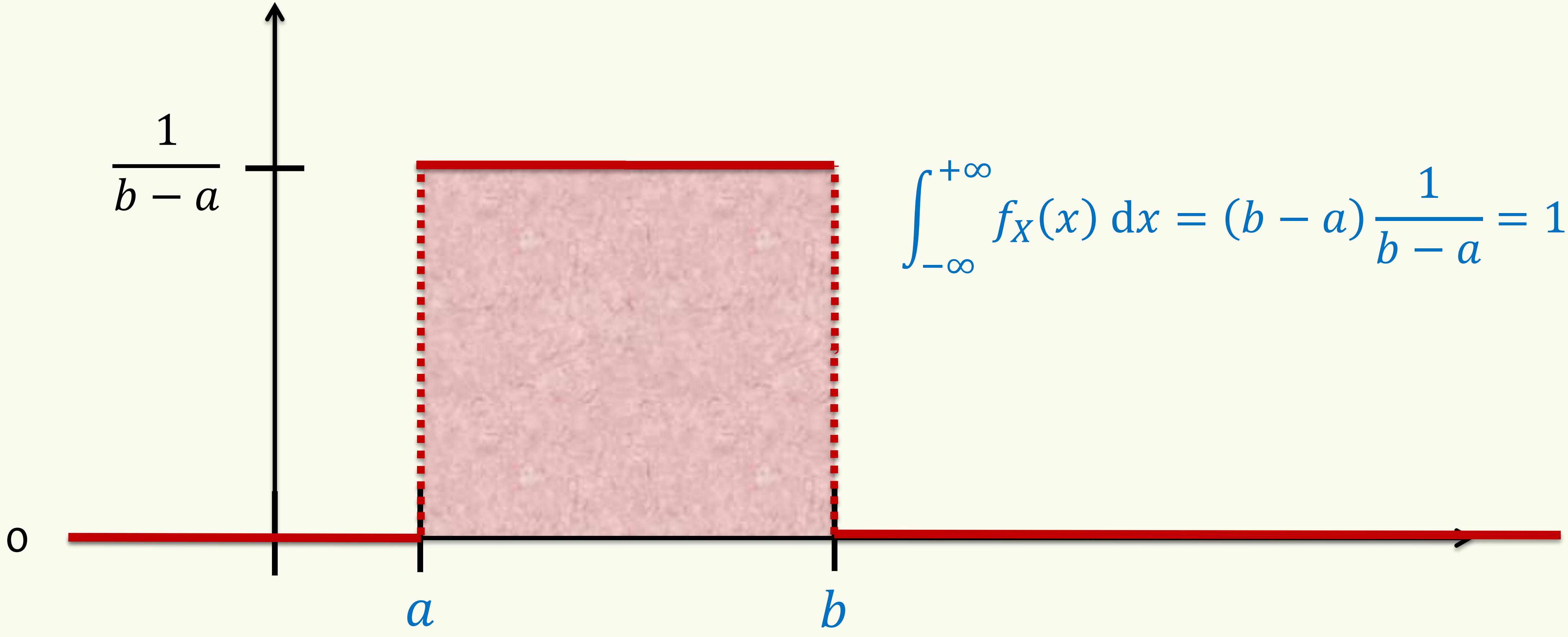
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Probability on $[0,0.5]$ accumulates at twice the rate compared to $\text{Unif}(0,1)$

Uniform Distribution

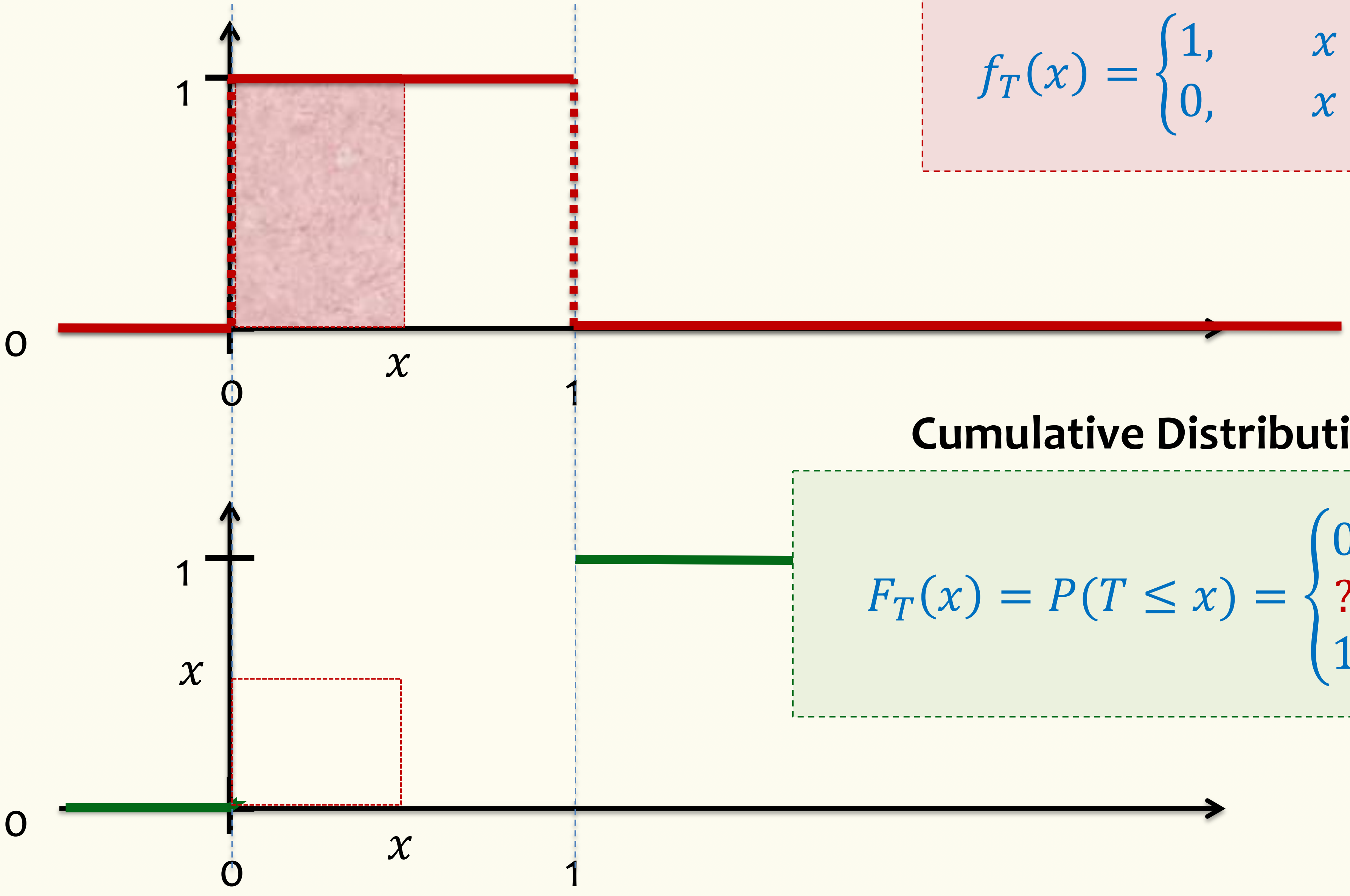
$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

Example. $T \sim \text{Unif}(0,1)$



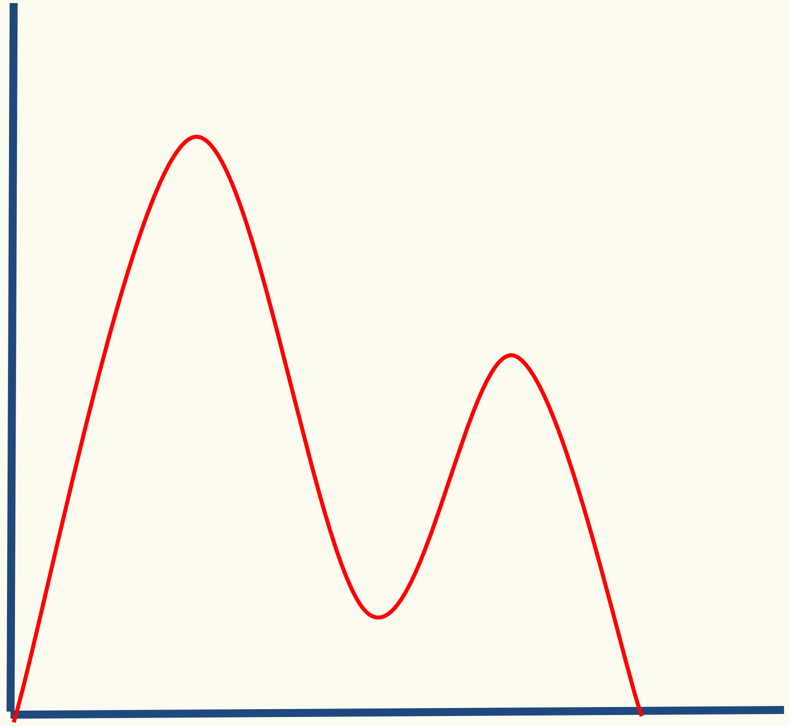
Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Cumulative Distribution Function

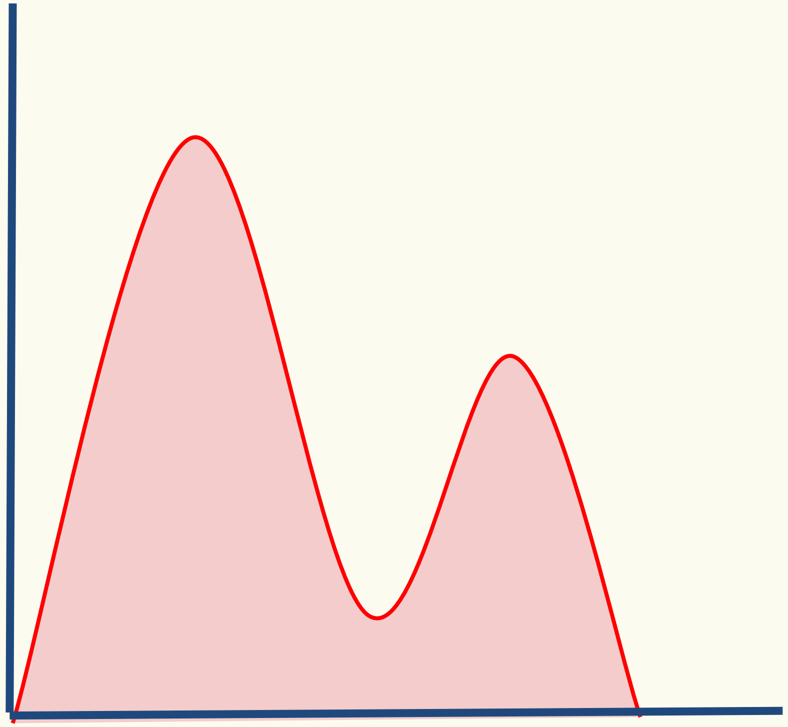
$$F_T(x) = P(T \le x) = \begin{cases} 0 & x \le 0 \\ ? & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

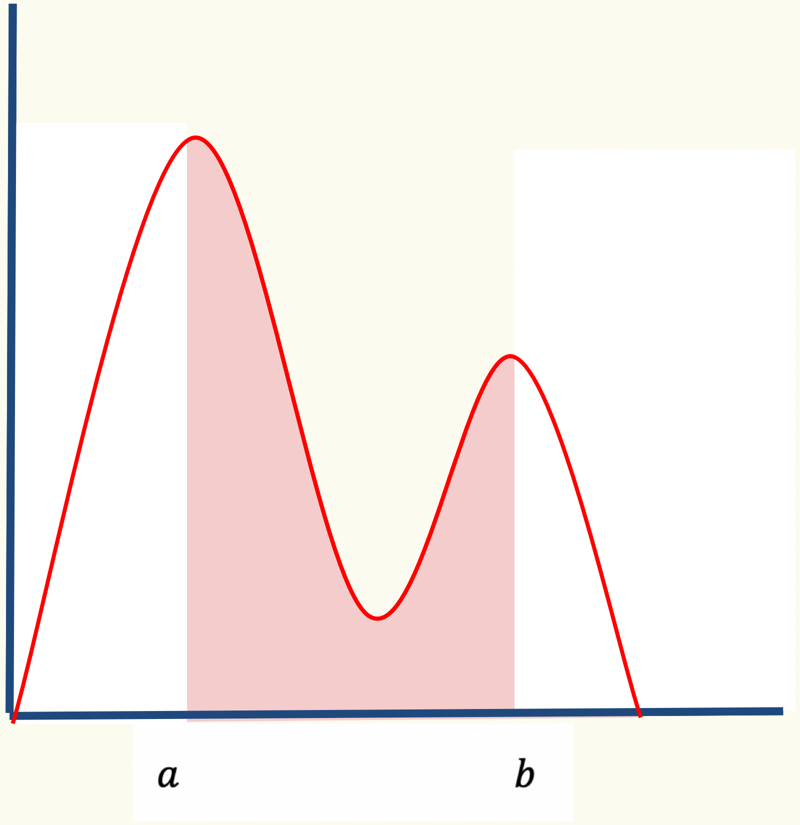
Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

Probability Density Function - Intuition

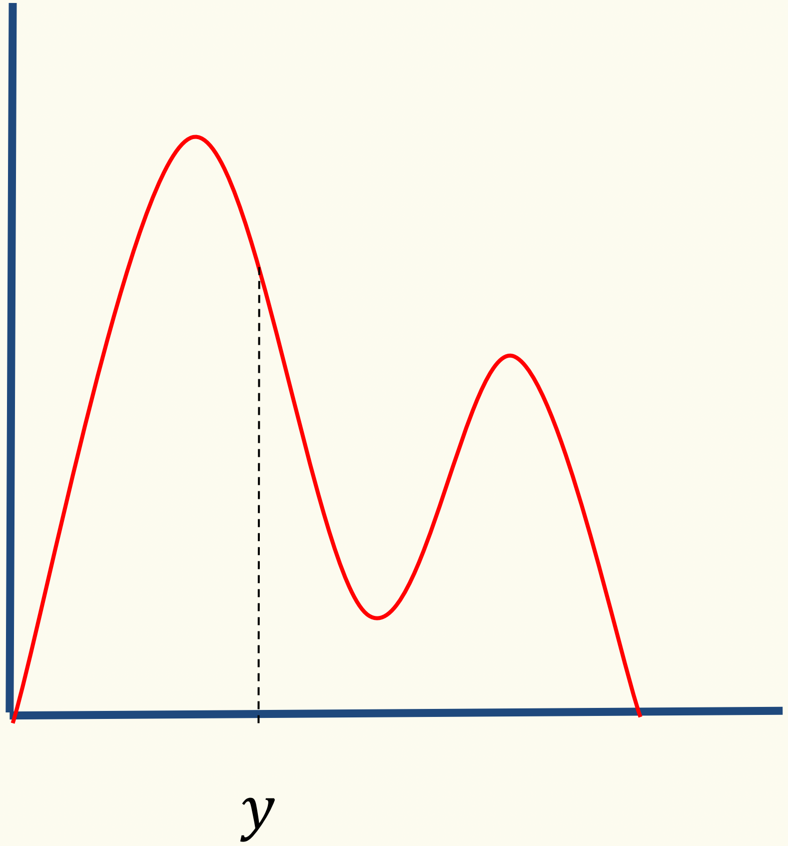


Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

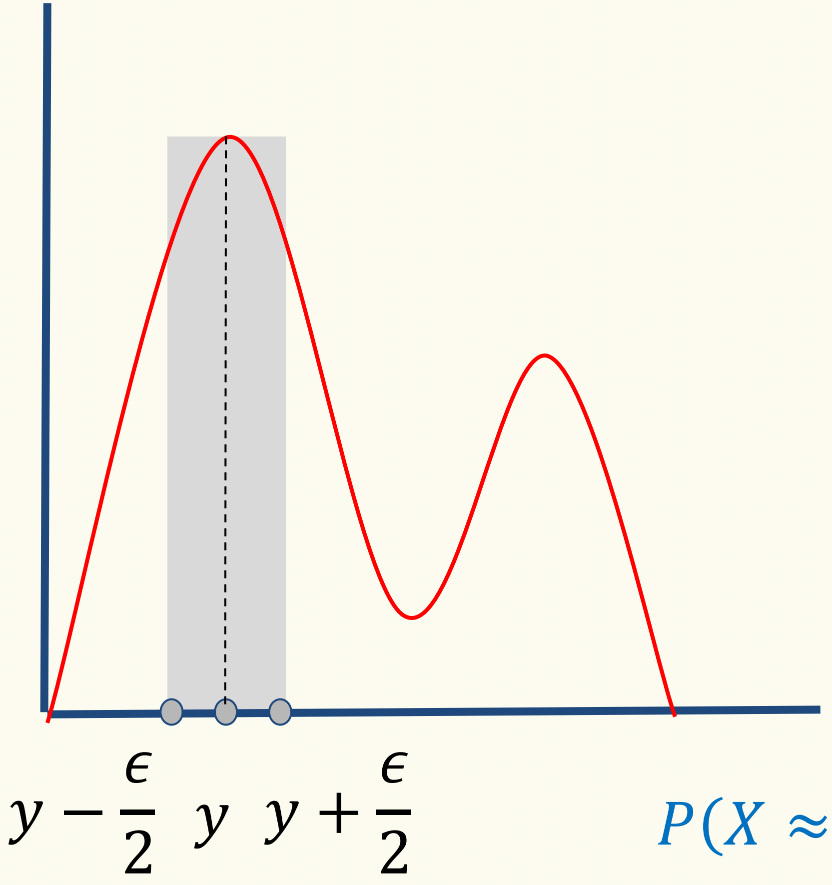
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

Density ≠ Probability

$f_X(y) \neq 0 \qquad P(X = y) = 0$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

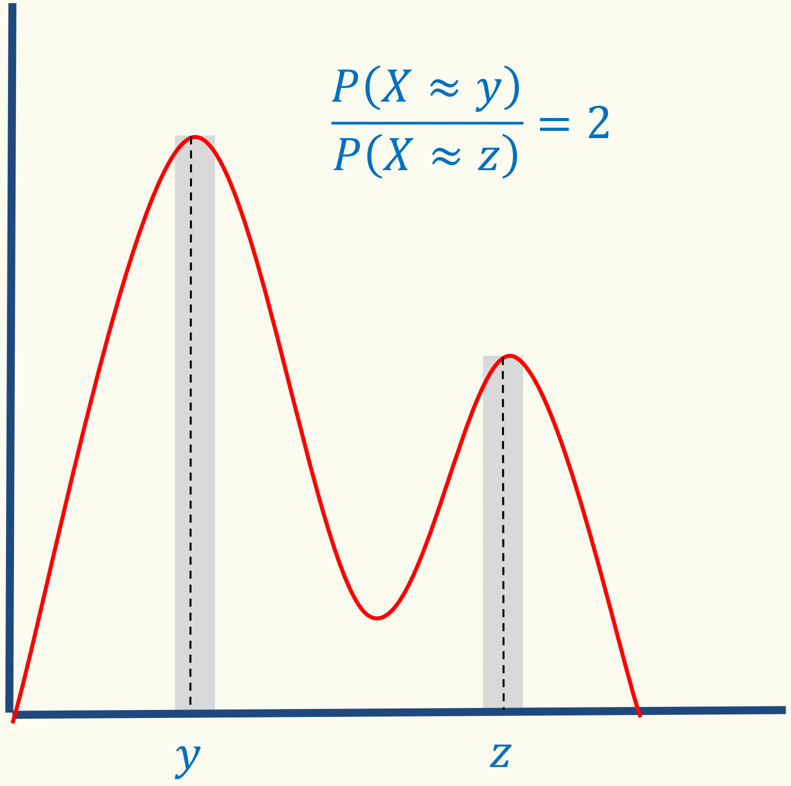
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

What $f_X(x)$ measures: The local ***rate*** at which probability accumulates

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = P(X \le a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

So: $f_X(x) = \frac{d}{dx} F_X(x)$

$$P(X \in [a,b]) = \int_{-\infty}^{b} f_X(x)\,\mathrm{d}x - \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x = F_X(b) - F_X(a)$$

F_X is monotone increasing, since $f_X(x) \ge 0$. That is $F_X(c) \le F_X(d)$ for $c \le d$

$\lim_{a \to -\infty} F_X(a) = P(X \le -\infty) = 0$ $\quad$ $\lim_{a \to +\infty} F_X(a) = P(X \le +\infty) = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$