CSE 312

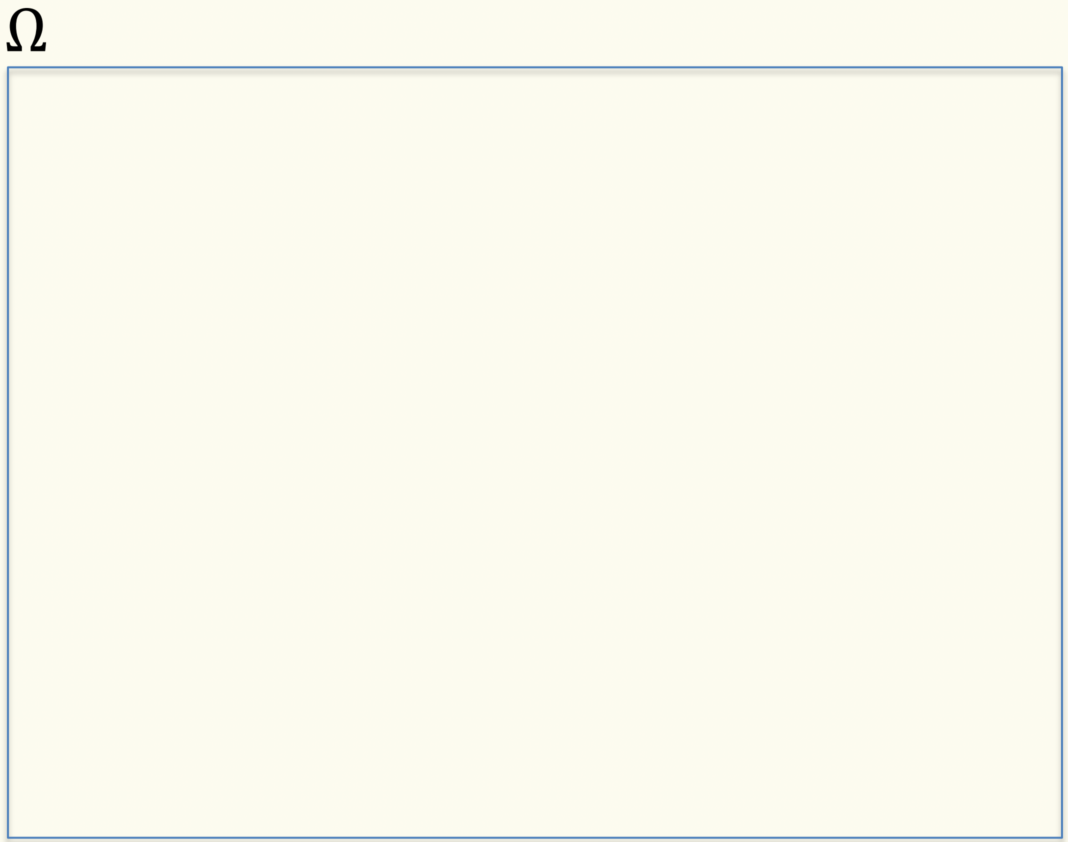
Foundations of Computing II

Lecture 7: Random Variables

Announcements

- PSet 1 graded + solutions on canvas
- PSet 2 due tonight
- Pset 3 posted this evening

Review



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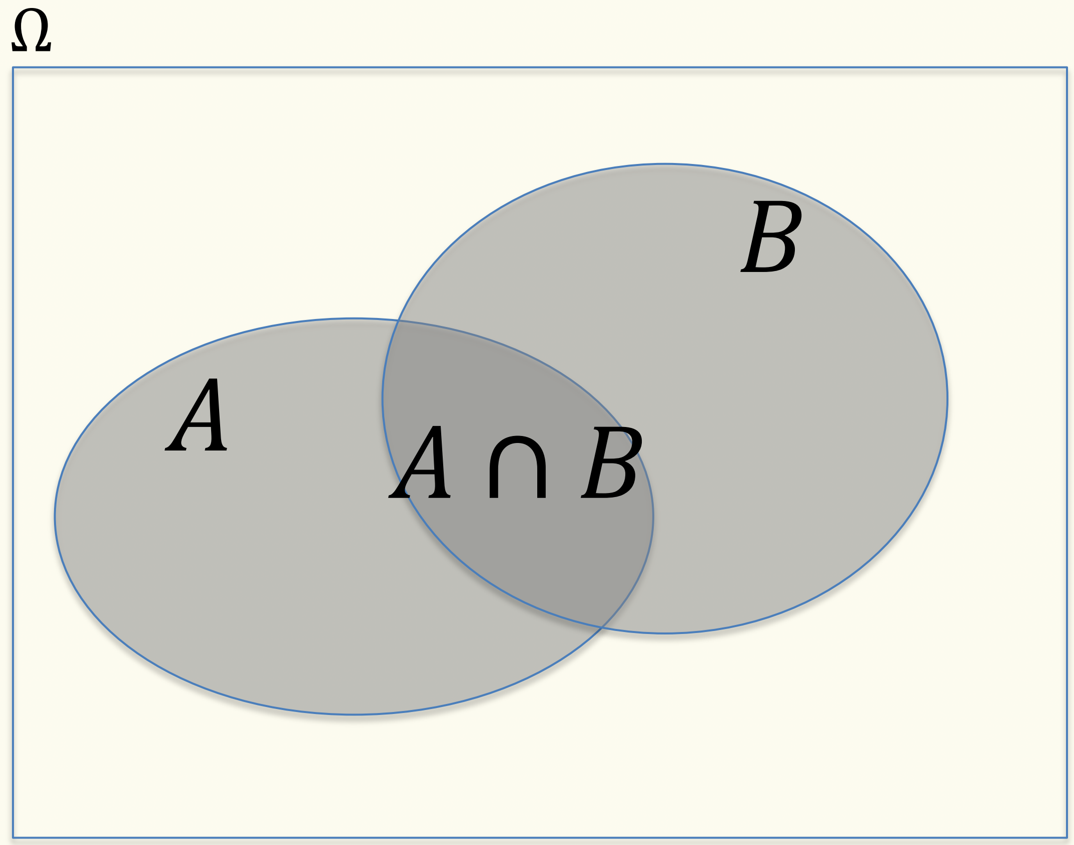
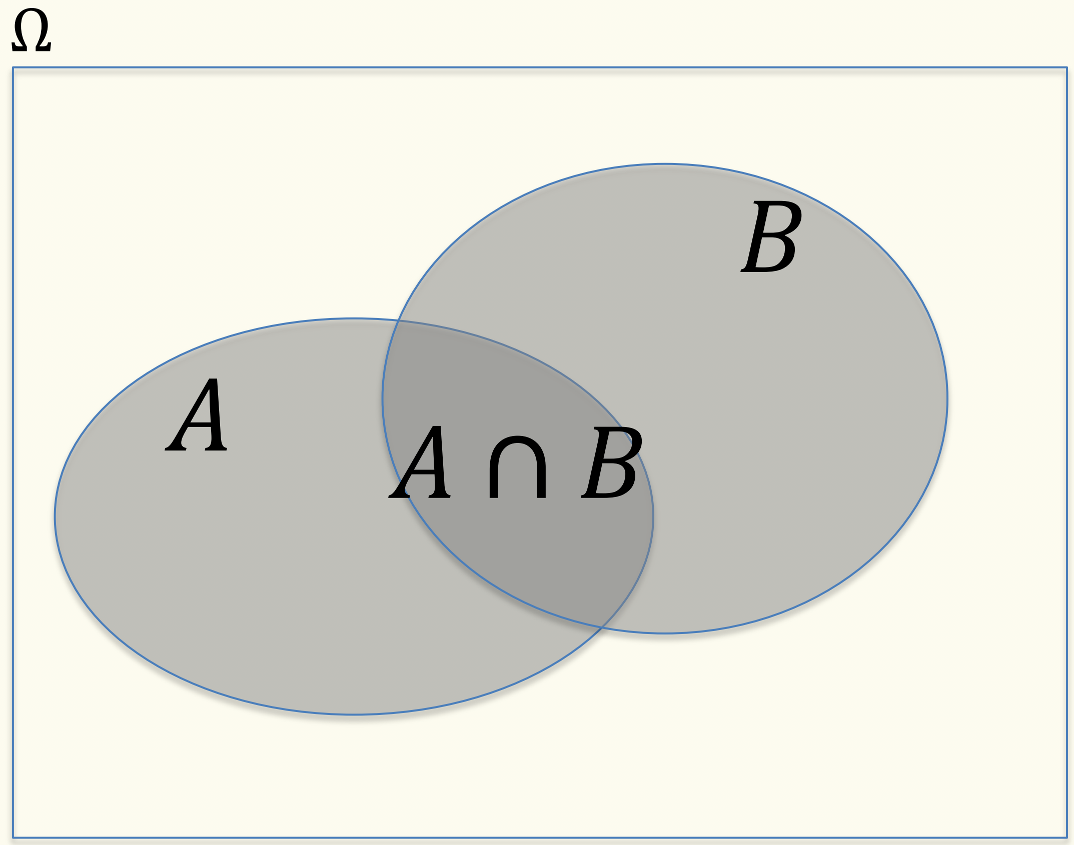


$$p(A)$$

Density of A in Ω.

$p(A|B)$ density of $A \cap B$ in B.

Review

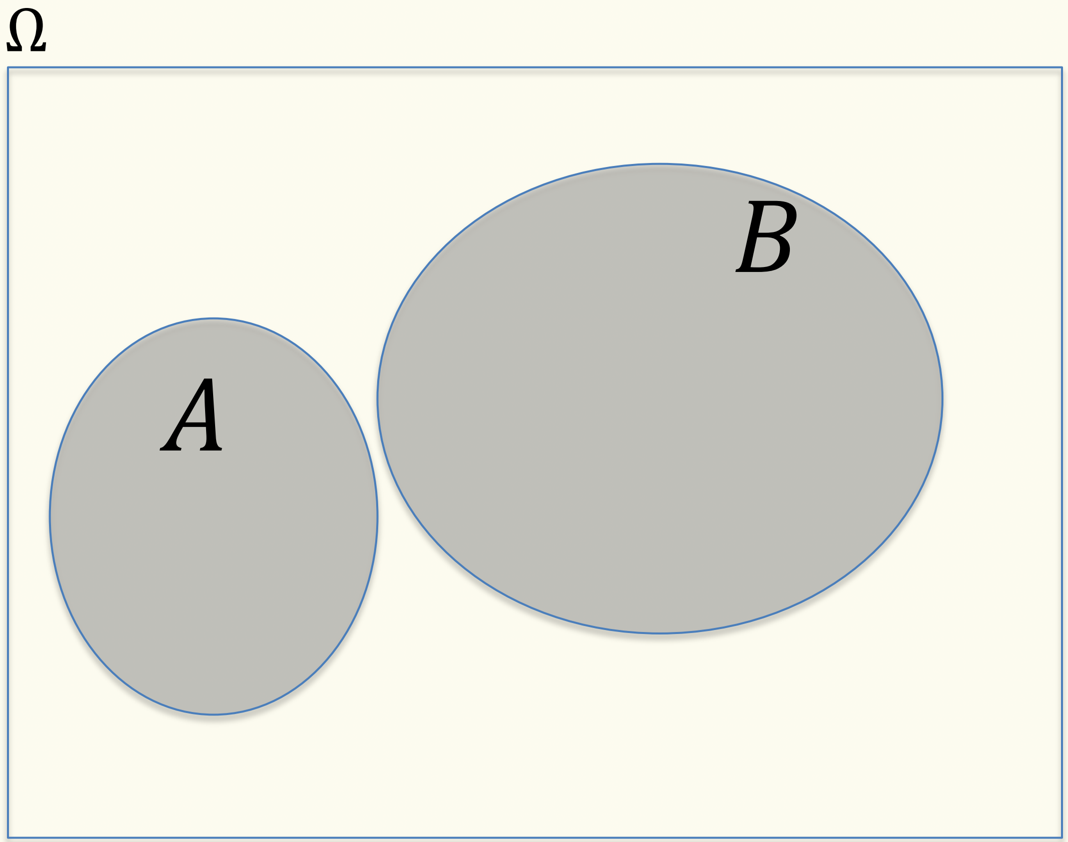


$p(A)$
Density of A in Ω.

$p(A|B)$ density of $A \cap B$ in B.

$$p(A \cap B) = p(B) \cdot p(A|B)$$

Review



$$p(A)$$

Density of A in Ω.

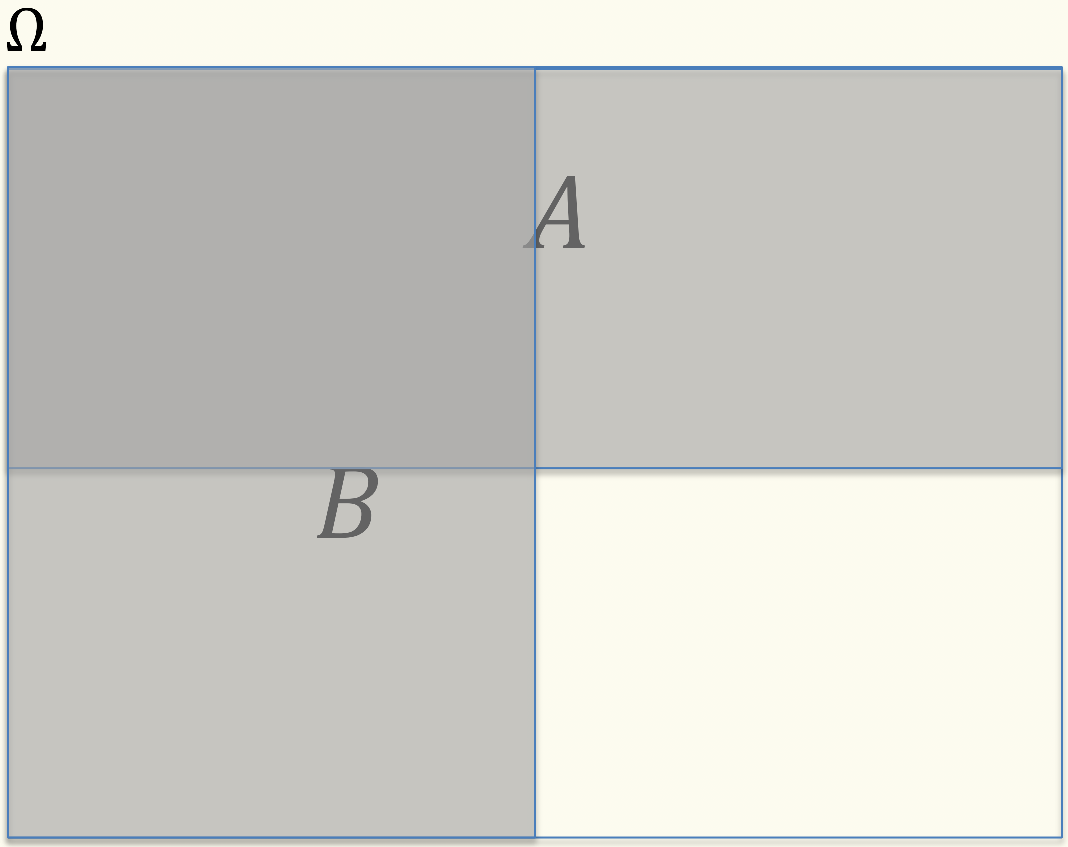
$p(A|B)$ density of $A \cap B$ in B.

$$p(A \cap B) = 0$$

"mutually exclusive"

Review

$p(A|B)$ density of $A \cap B$ in B.



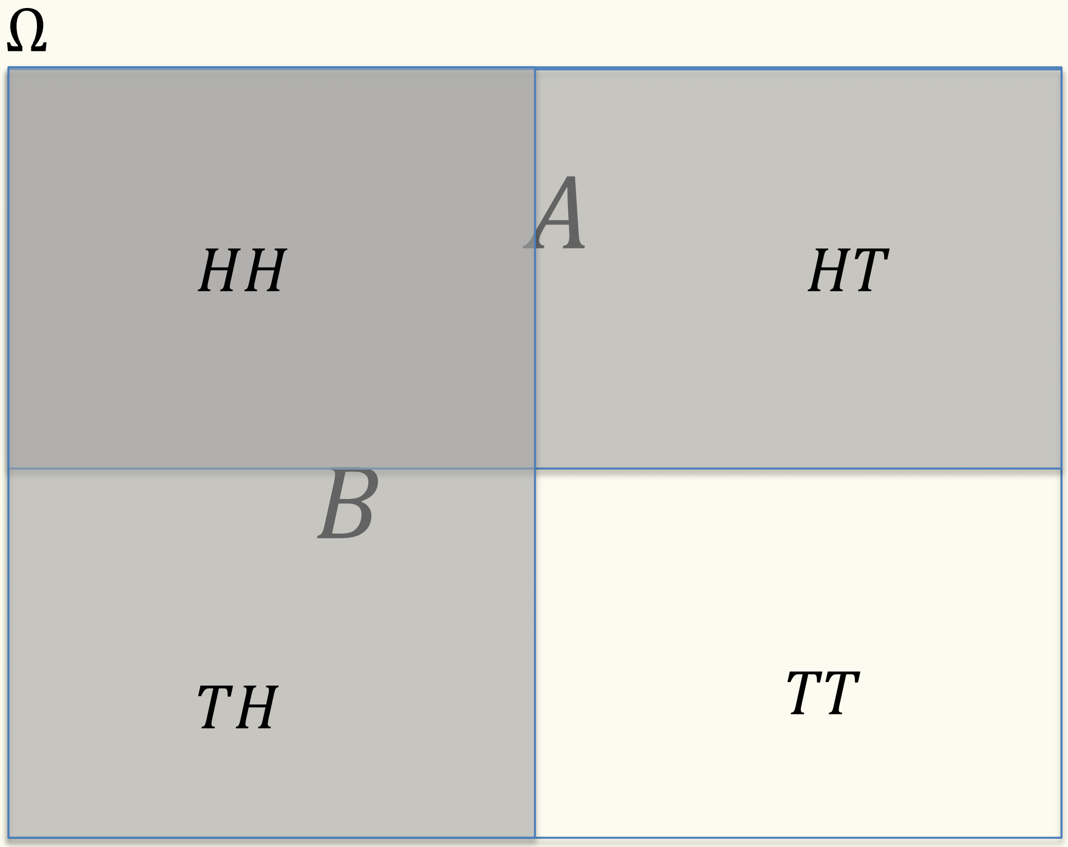
$p(A \cap B) = p(B) \cdot p(A|B) =$
$p(B) \cdot p(A)$.

"mutually independent"

Knowing B happened does not change the probability that A happened.

Review

$A: first\ toss\ is\ H$
$B: second\ toss\ is\ H$



$p(A|B)$ density of $A \cap B$ in B.

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$p(B) \cdot p(A)$.
"mutually independent"

Knowing B happened does not change the probability that A happened.

Review Chain rule & independence

Theorem. (Chain Rule) For events $A_1, A_2, \dots, A_n$,

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

Definition. Two events A and A are **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

"Equivalently." $P(A|B) = P(A)$.

One more related item: Conditional Independence

Definition. Two events A and B are **independent** conditioned on C if

$P(C) \neq 0$ and $P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$.

- If $P(A \cap C) \neq 0$, equivalent to $P(B|A \cap C) = P(B \mid C)$
- If $P(B \cap C) \neq 0$, equivalent to $P(A|B \cap C) = P(A \mid C)$

Plain Independence. Two events A and B are **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

- If $P(A) \neq 0$, equivalent to $P(B|A) = P(B)$
- If $P(B) \neq 0$, equivalent to $P(A|B) = P(A)$

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

Random Variables

Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its *range/support*

Two common notations: $X(\Omega)$ or Ω_X

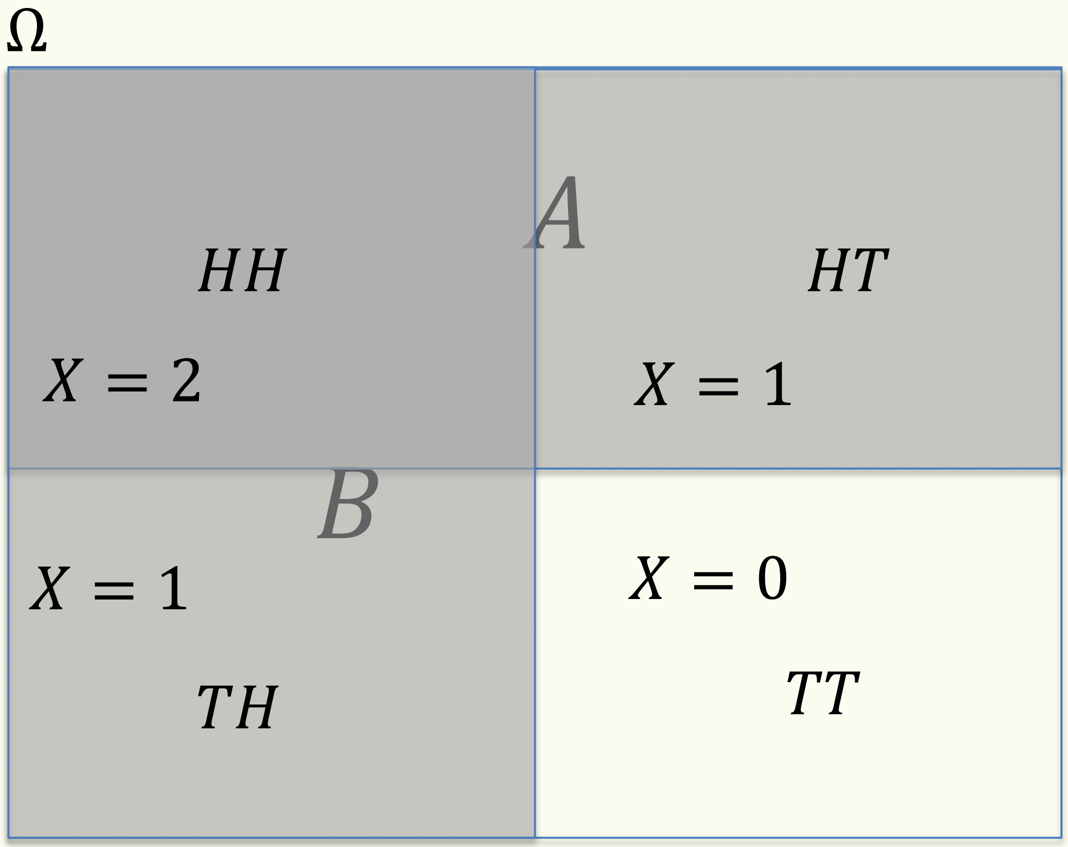
Example. Two coin flips: $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

$X =$ number of heads in two coin flips

$X(\mathrm{HH}) = 2 \quad X(\mathrm{HT}) = 1 \quad X(\mathrm{TH}) = 1 \quad X(\mathrm{TT}) = 0$

range (or support) of X is $X(\Omega) = \{0,1,2\}$

Review



A: first toss is H

B : second toss is H

X : number of heads

$p(A|B)$ density of $A \cap B$ in B.

$p(A \cap B) = p(B) \cdot p(A|B) =$
$p(B) \cdot p(A)$.
"mutually independent"

Knowing B happened does not change the probability that A happened.

Another RV Example

20 different balls labeled 1, 2, … , 20 in a jar

- Draw a subset of 3 from the jar uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$
  - Example: $X(\{2, 7, 5\}) = 7$
  - Example: $X(\{15, 3, 8\}) = 15$

Random Variables

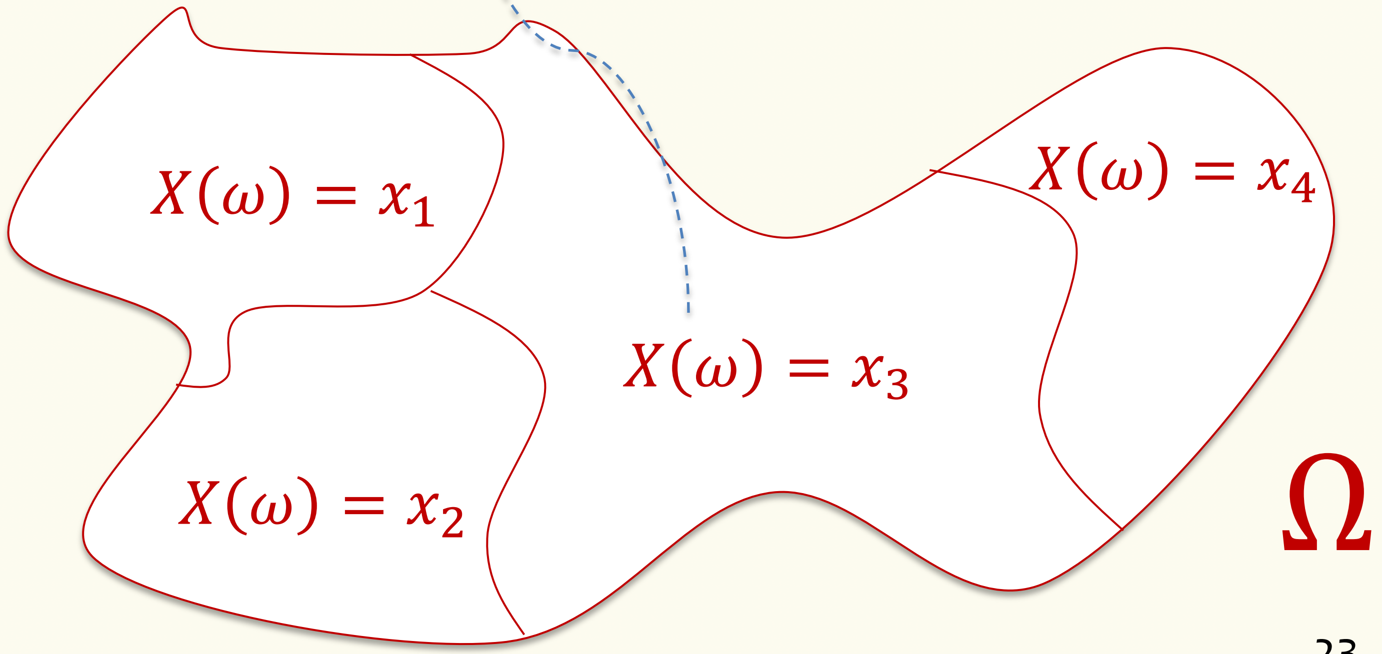
Definition. For a RV $X : \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $P(X = x) = P(\{X = x\})$

Random variables **partition** the sample space.

$\Sigma_{x \in X(\Omega)} P(X = x) = 1$



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Example. Two coin flips: $\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

$X =$ number of heads in two coin flips $\qquad \Omega_X = X(\Omega) = \{0,1,2\}$

$$P(X = 0) = \frac{1}{4} \qquad P(X = 1) = \frac{1}{2} \qquad P(X = 2) = \frac{1}{4}$$

The RV X yields a new probability distribution with sample space $\Omega_X \subset \mathbb{R}$!

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Probability Mass Function (PMF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the function $p_X: \Omega_X \to \mathbb{R}$ defined by $p_X(x) = P(X = x)$ is called the **probability mass function (PMF)** of X

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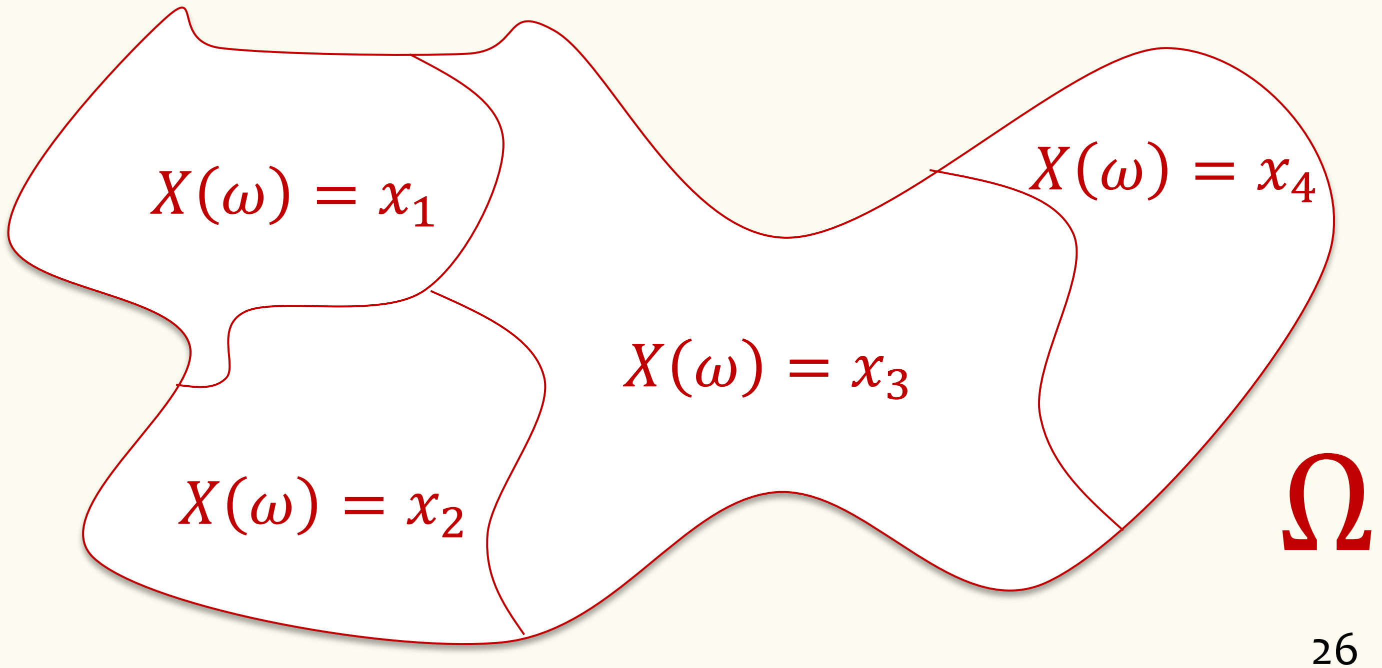

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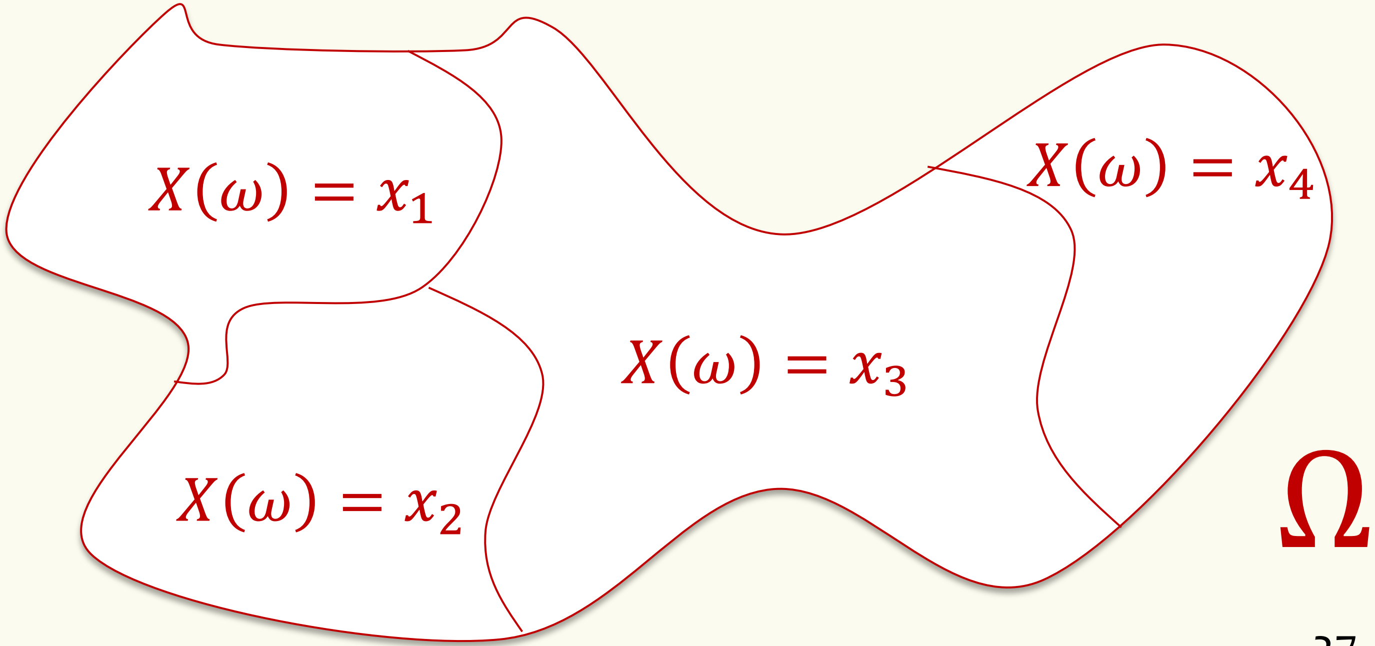

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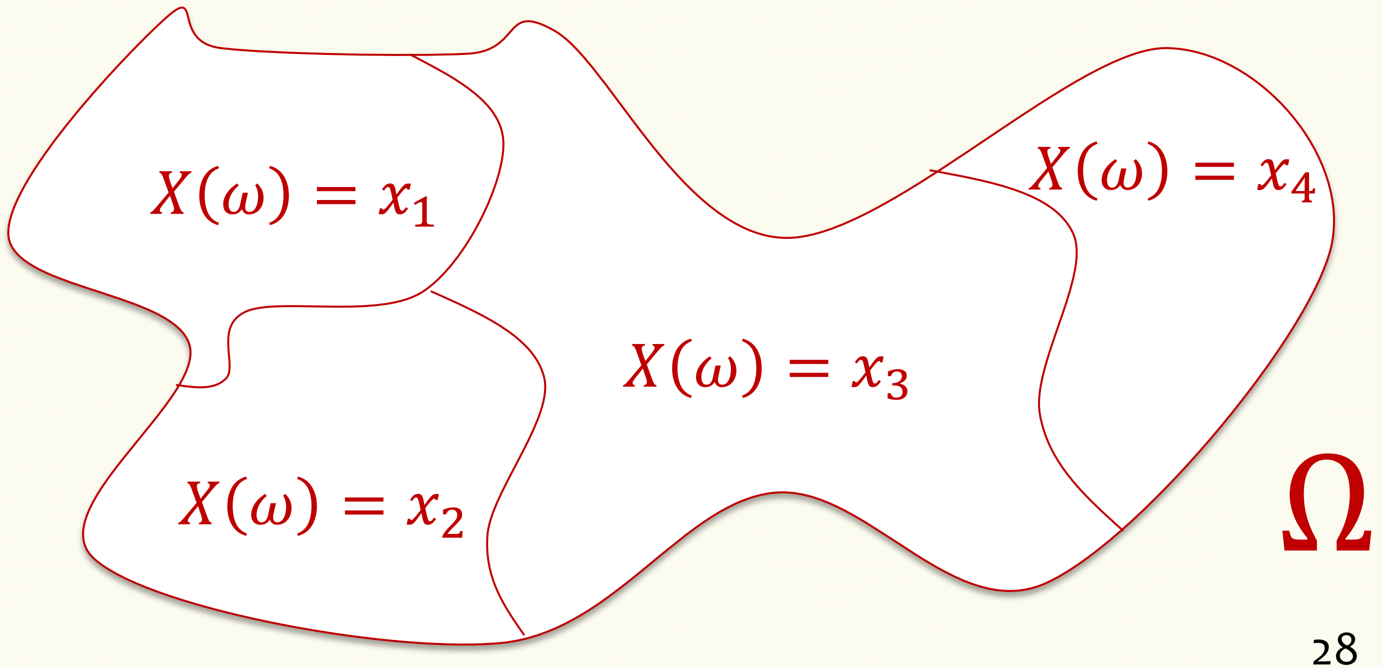
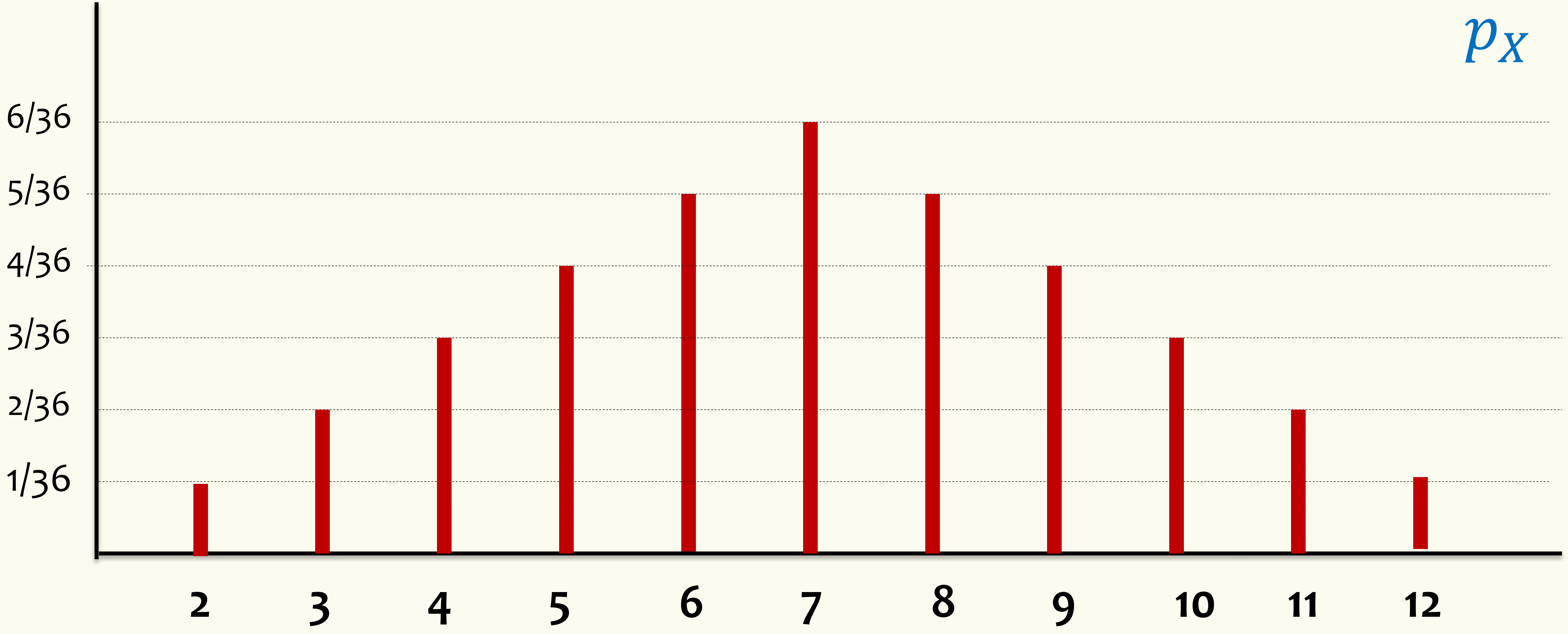


Example – Two Fair Dice

X = sum of two dice throws



Example – Number of Heads

We flip n coins, independently, each heads with probability p

$$\Omega = \{\mathrm{HH}\cdots\mathrm{HH}, \mathrm{HH}\cdots\mathrm{HT}, \mathrm{HH}\cdots\mathrm{TH}, \ldots, \mathrm{TT}\cdots\mathrm{TT}\}$$

X = # of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

# of sequences with k heads

Prob of sequence w/ k heads

Example – Number of Heads

We flip n coins, independently, each heads with probability p

$\Omega = \{\mathrm{HH} \cdots \mathrm{HH}, \mathrm{HH} \cdots \mathrm{HT}, \mathrm{HH} \cdots \mathrm{TH}, \dots, \mathrm{TT} \cdots \mathrm{TT}\}$

$X =$ # of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$\sum_{k=0}^{n} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} =$$

Example – Number of Heads

We flip n coins, independently, each heads with probability p

$\Omega = \{\mathrm{HH}\cdots\mathrm{HH}, \mathrm{HH}\cdots\mathrm{HT}, \mathrm{HH}\cdots\mathrm{TH}, \ldots, \mathrm{TT}\cdots\mathrm{TT}\}$

$X =$ # of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

$$\sum_{k=0}^{n} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + 1 - p)^k = 1.$$

Binomial theorem

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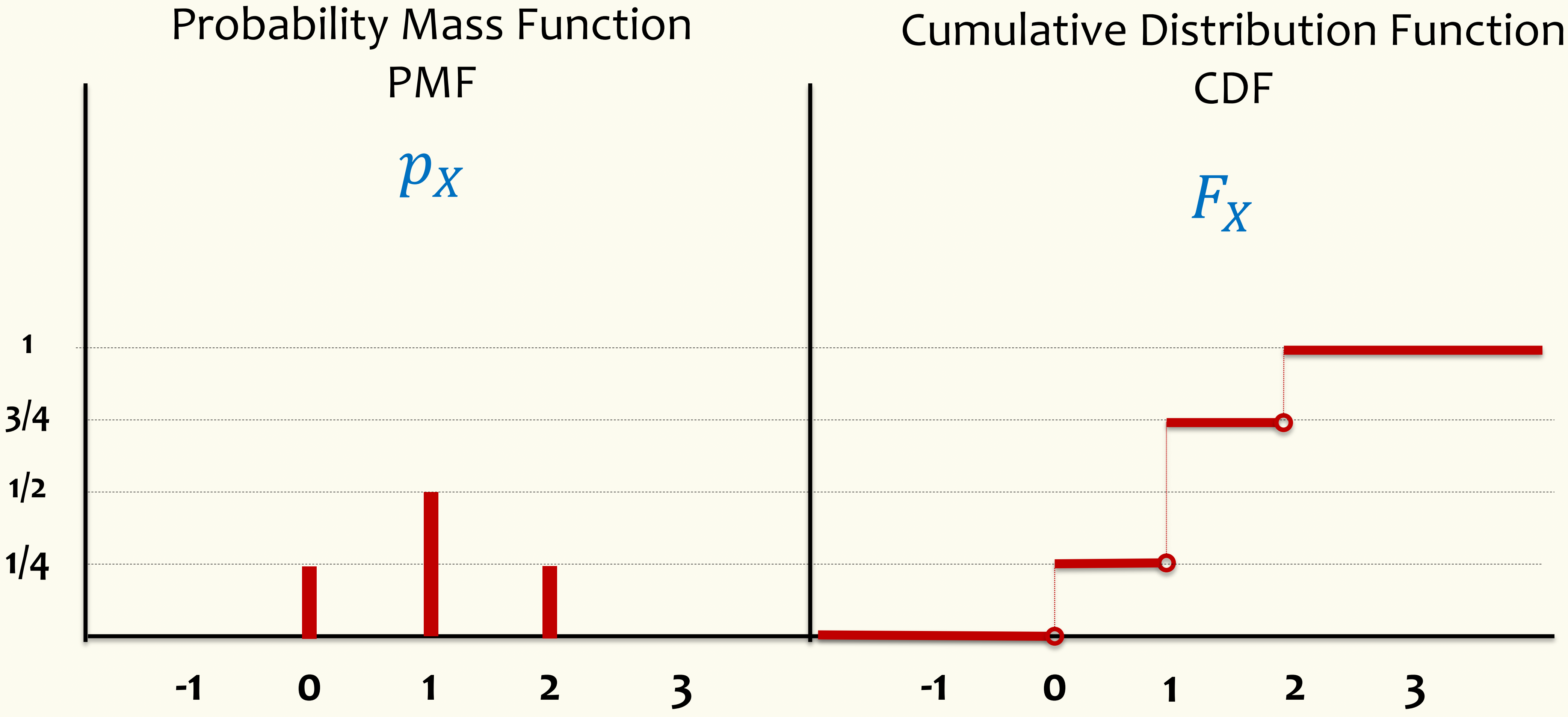
Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of X is the function $F_X: \mathbb{R} \to [0,1]$ that specifies for any real number x, the probability that $X \leq x$.

That is, F_X is defined by $F_X(x) = P(X \leq x)$

Example – Two fair coin flips

X = number of heads



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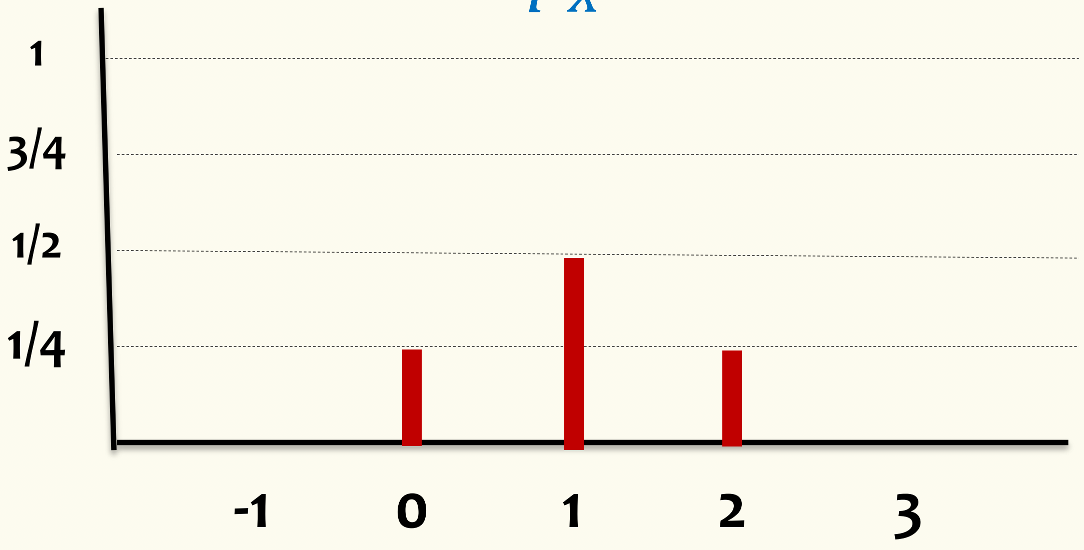
Expectation (Idea)

Example. Two fair coin flips

$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

X = number of heads

p_X



- If we chose samples from Ω over and over repeatedly, how many heads would we expect to see per sample from Ω?
  - The idealized number, not the average of actual numbers seen (which will vary from the ideal)

Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \mathrm{X}(\Omega)} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Expected Value

Definition. The expected value of a (discrete) RV X is

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x) = \sum_x x \cdot P(X = x)$$

Example. Value X of rolling one fair die

$$p_X(1) = p_X(2) = \cdots = p_X(6) = \frac{1}{6}$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

For the equally-likely outcomes case, this is just the average of the possible outcomes!

Expectation

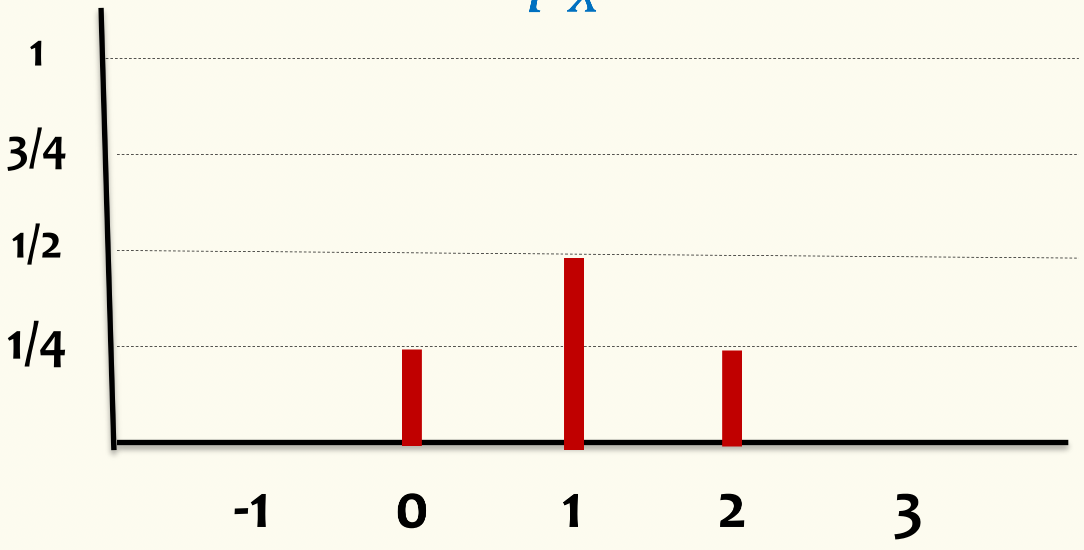
Example. Two fair coin flips

$$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$$

$$X = \text{number of heads}$$

What is $\mathbb{E}[X]$?

p_X



$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, <u>on average</u>, per game, when repeatedly playing?"

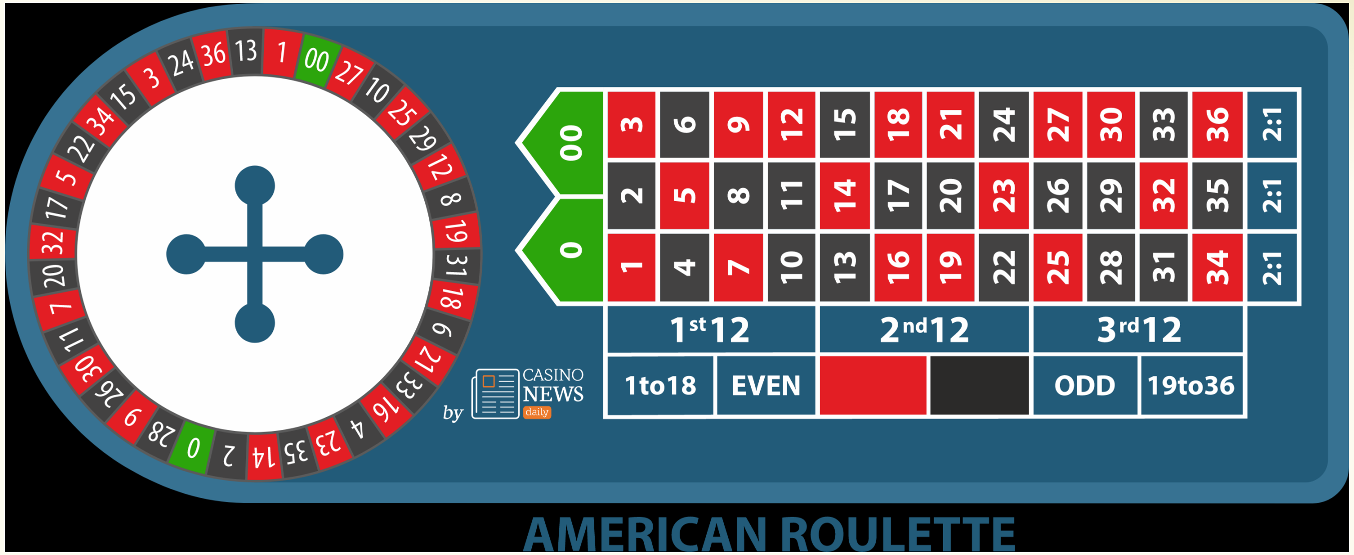
Answer: $\mathbb{E}[X]$

Roulette (USA)

Ω:

Numbers 1-36
- 18 Red
- 18 Black

Green 0 and 00



Note 0 and 00 are not EVEN

RVs for gains from some bets:

RV RED: If Red number turns up $+1$, if Black number, 0, or 00 turns up -1

$$\mathbb{E}[\text{RED}] = (+1)\cdot\frac{18}{38} + (-1)\cdot\frac{20}{38} = -\frac{2}{38} \approx -5.26\%$$

RV 1st12: If number 1-12 turns up $+2$, if number 13-36, 0, or 00 turns up -1

$$\mathbb{E}[1^{st}12] = (+2)\cdot\frac{12}{38} + (-1)\cdot\frac{26}{38} = -\frac{2}{38} \approx -5.26\%$$

Roulette (USA)

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Numbers 1-36

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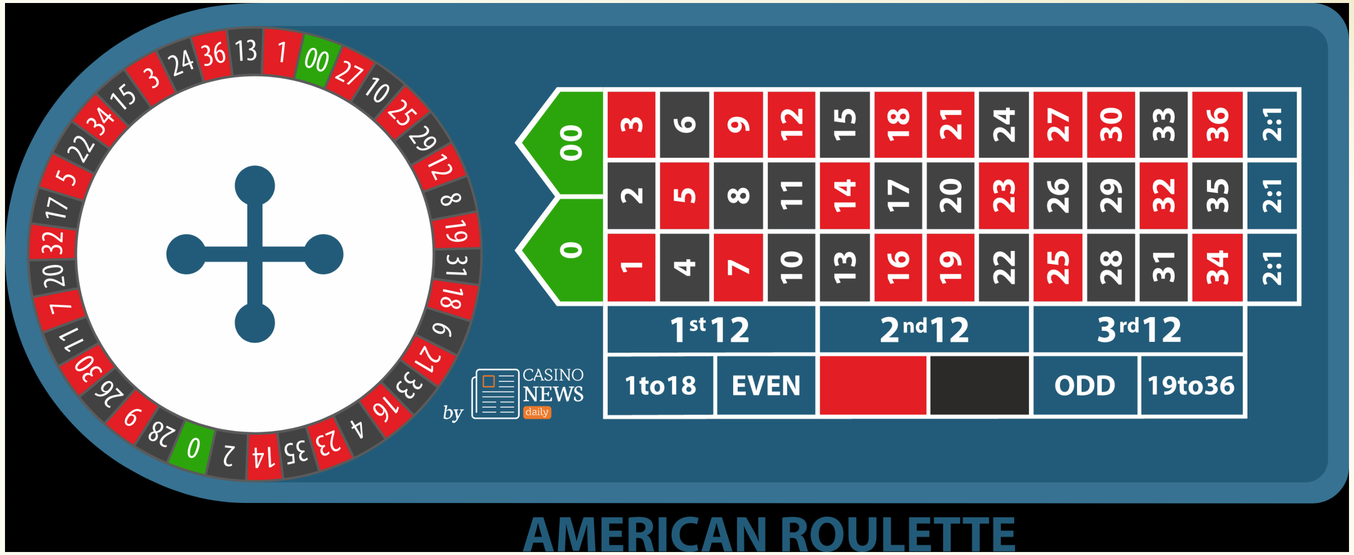

Note 0 and 00 are not EVEN

An even worse bet:

RV BASKET: If 0, 00, 1, 2, or 3 turns up $+6$ otherwise -1

$$\mathbb{E}[\text{BASKET}] = (+6)\cdot\frac{5}{38} + (-1)\cdot\frac{33}{38} = -\frac{3}{38} \approx -7.89\%$$

Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
$1/6$	$1, 2, 3$	3
$1/6$	$1, 3, 2$	1
$1/6$	$2, 1, 3$	1
$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= 6 \cdot \frac{1}{6} = 1$$

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Next time: Properties of Expectation