CSE 312

Foundations of Computing II

Lecture 3: Even more counting

Binomial Theorem, Inclusion-Exclusion, Pigeonhole Principle

Recap

Two core rules for counting a set S:

- Sum rule:
  - Break up S into disjoint pieces/cases
  - $|S|$ = the sum of the sizes of the pieces.

- Product rule:
  - View the elements of S as being constructed by a series of choices, where the # of possibilities for each choice doesn't depend on the previous choices
  - $|S|$ = the product of the # of choices in each step of the series.

Recap

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule → n^k

- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation → $\frac{n!}{(n-k)!}$

- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination → $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Coefficients – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Symmetry in Binomial Coefficients

Fact. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Follows from Binomial Theorem

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Identity

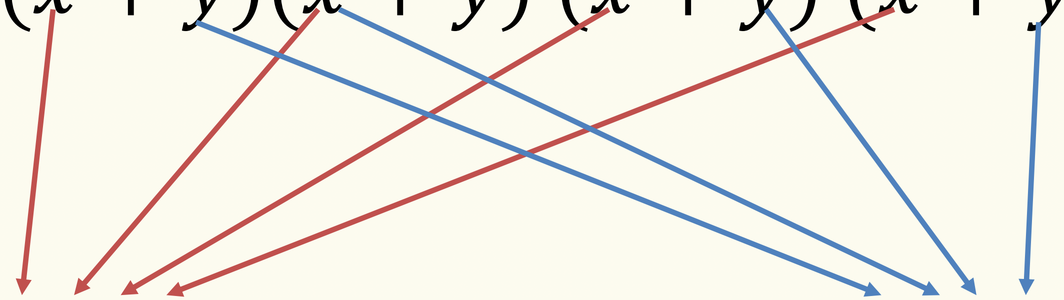
Binomial Theorem: Idea

$$(x+y)^2 = (x+y)(x+y)$$

$$= xx + xy + yx + yy$$

$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x+y)^n = (x+y)\dots(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x+y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x+y)$ (the other $n-k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Apply with $x = y = 1$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Apply with $x = 1, y = -1$

Corollary.

$$\binom{\boldsymbol{n}}{\mathbf{0}} - \binom{\boldsymbol{n}}{\mathbf{1}} + \binom{\boldsymbol{n}}{\mathbf{2}} - \cdots = \mathbf{0}$$

Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ ...$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

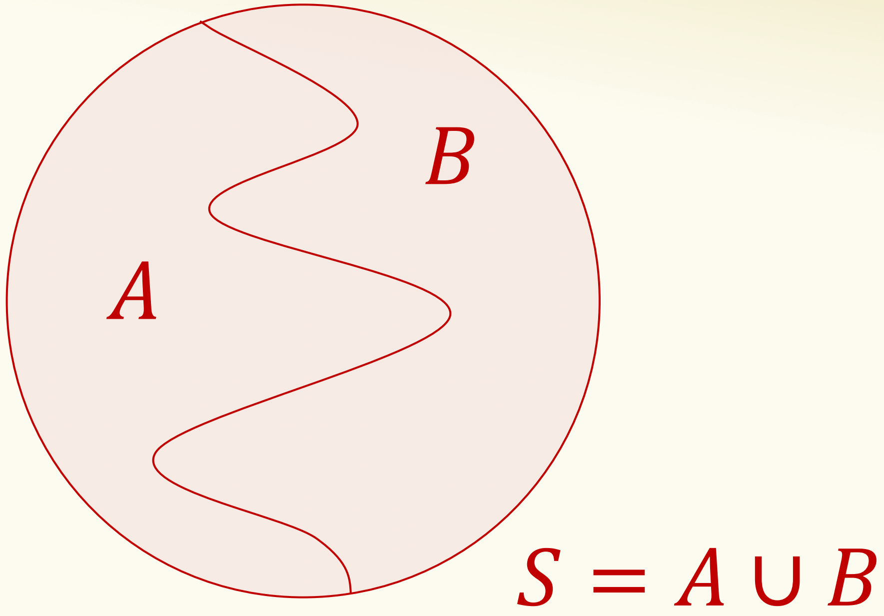
Hard work and not intuitive

Let's see a combinatorial argument

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

$|S| = \binom{n}{k}$

e.g. $n = 4, k = 2$, $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n]$ that DO include n

$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$

B: set of size k subsets of $[n]$ that DON'T include n

$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

n is in set, need to choose other $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

A: set of size k subsets of $[n]$ that DO include n

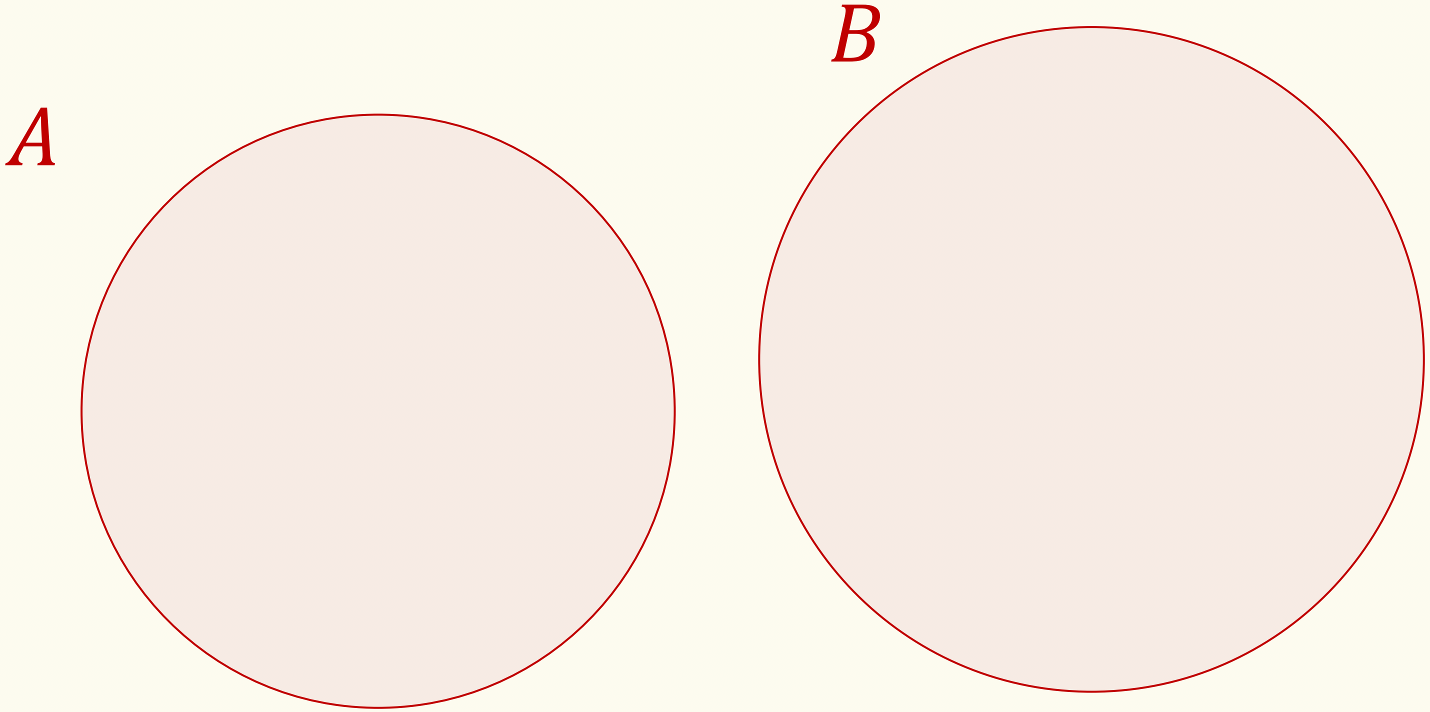
n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

B: set of size k subsets of $[n]$ that DON'T include n

Recap Disjoint Sets

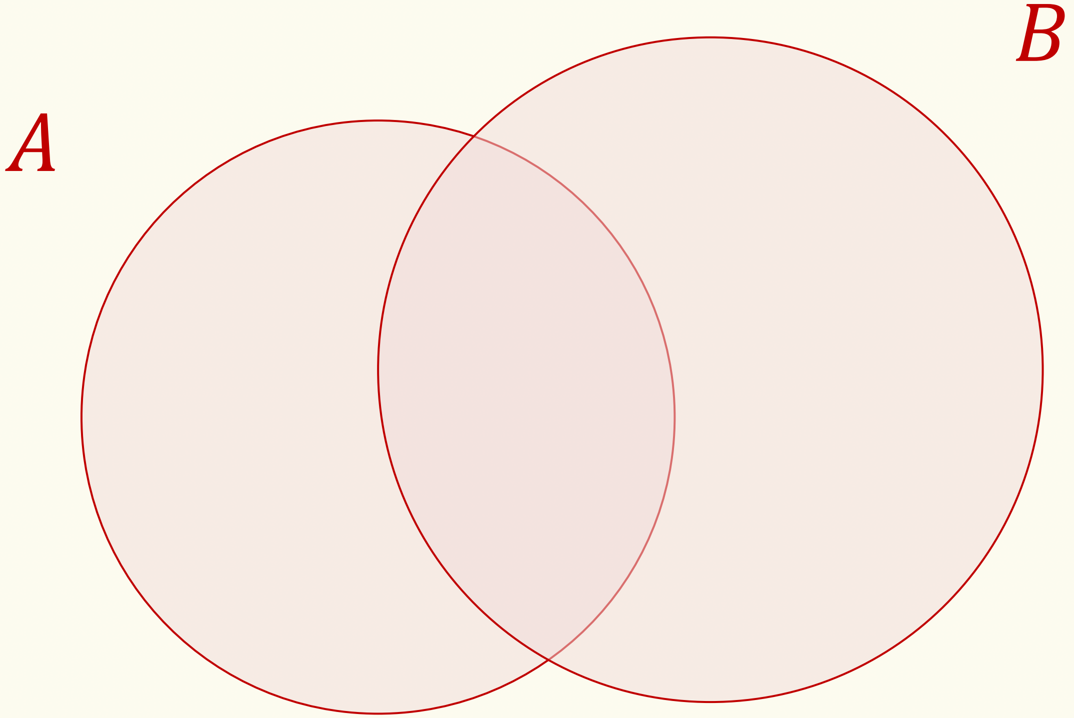
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?

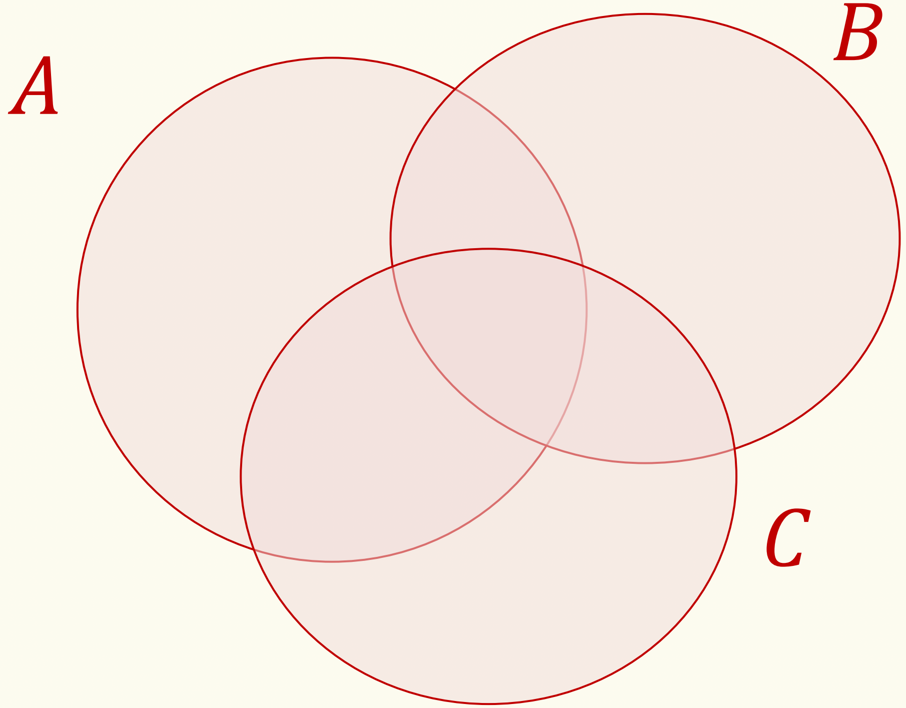


$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

What if there are three sets?

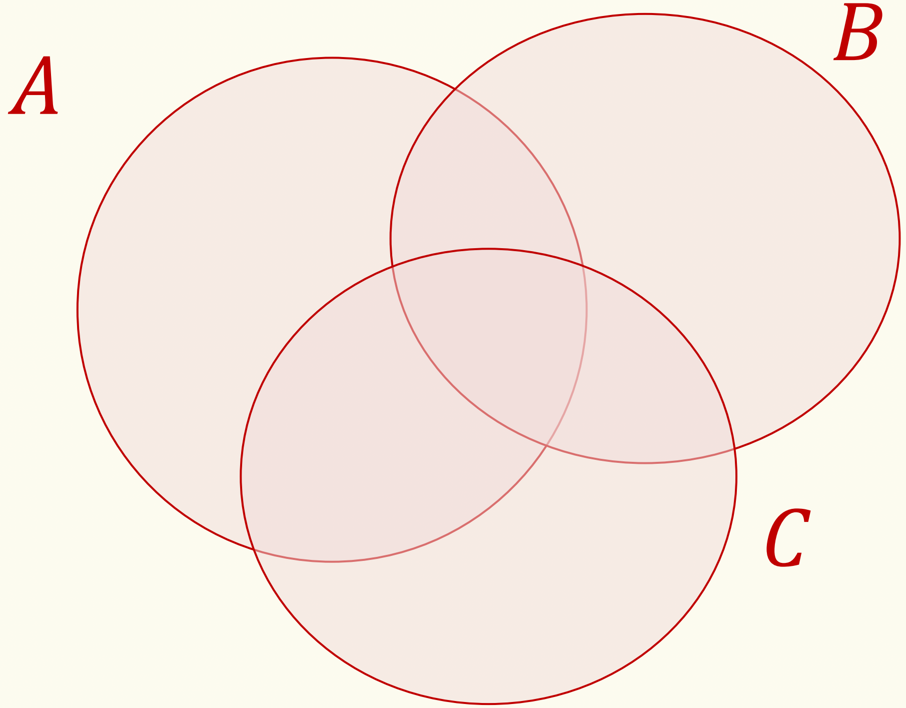


Fact.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

What if there are three sets?



Fact.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Pf:

Consider any element $x \in A \cup B \cup C$.
Suppose x is in k sets.
Then x contributes:

$$\binom{k}{1} - \binom{k}{2} + \cdots \binom{k}{k}$$

by Binomial theorem, $= \binom{k}{0} = 1$.

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = singles\ - doubles + triples\ - quads + \ \ldots$$
$$= (|A_1| + \cdots + |A_n|)\ - (|A_1 \cap A_2| + \ \ldots + |A_{n-1} \cap A_n|) + \ \ldots$$

Brain Break

Pigeonhole Principle (PHP): Idea

10 pigeons, 9 holes

At least one hole must get 2 pigeons!

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k \cdot \frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **366** holes (365 for a normal year + Feb 29) = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
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Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeons: integers x in S

Pigeonholes: $\{0,1,\ldots,36\}$

Assignment: x goes to $x \bmod 37$

Since $100 > 37$, by PHP, there are $x \neq y \in S$ s.t.
$x \bmod 37 = y \bmod 37$ which implies
$x - y = 37\,k$ for some integer k

Pigeonhole Principle – Example

In every sequence of n numbers, there must be either an increasing subsequence of $\sqrt{n}$ numbers or a decreasing sequence of $\sqrt{n}$ numbers.

Example:

1,2,3,**4,5,6**,7,8,9

9,8,7,6,5,4,3,2,1

3,4,2,1,7,9,8,5,6

Given $x_1, x_2, \dots, x_n$.

Suppose longest increasing subseq is of length I, longest decreasing subseq is of length D.

Pigeons: $\{1, \dots, n\}$. Holes: $\{1, \dots, I\} \times \{1, \dots, D\}$.

Put pigeon i in hole (a, b) if a is longest inc. subseq. ending at x_i, b is longest dec. subseq. ending at x_i.

Claim: $i < j$ cannot be mapped to the same hole!
Pf: If $x_i \leq x_j$, longest inc subseq ending at x_i can be extended by adding x_j. So, length of longest inc subseq cannot be same for x_i,x_j.
Similarly, if $x_i \geq x_j$ …

Given $x_1, x_2, \ldots, x_n$.

Suppose longest increasing subseq is of length I, longest decreasing subseq is of length D.

Pigeons: $\{1, \ldots, n\}$. Holes: $\{1, \ldots, I\} \times \{1, \ldots, D\}$.

Put pigeon i in hole (a, b) if a is longest inc. subseq. ending at x_i, b is longest dec. subseq. ending at x_i.

Claim: $i < j$ cannot be mapped to the same hole.

Then we must have $I \cdot D \geq n$, so either $I \geq \sqrt{n}$ or $\mathrm{D} \geq \sqrt{n}$.