CSE 312

Foundations of Computing II

Lecture 2: Combinations and Binomial Coefficients

Announcements

Homework:

- Pset1 was posted on Wednesday and is due 11:59pm next Wednesday.

Announcements

- **EdStem discussion etiquette**
  - OK to publicly discuss content of the course and any confusion over topics discussed in class, but **not *solutions*** for current homework problems, or anything about current exams that have not yet been graded.
  - It is also acceptable to ask for clarifications about what current homework problems are asking and concepts behind them, just not about their solutions.

Quick counting summary from last class

- Sum rule:

  If you can choose from

  – EITHER one of n options,

  – OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- Product rule:

  In a sequential process, if there are

  – n_1 choices for the 1^{st} step,

  – n_2 choices for the 2^{nd} step (given the first choice), …, and

  – n_k choices for the k^{th} step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times n_3 \times \cdots \times n_k$

- Representation of the problem is important (creative part)

Factorial

"How many ways to order elements in S, where $|S| = n$?"

Permutations

Answer = $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

Definition. The **factorial function** is

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Note: $0! = 1$

Theorem. (Stirling's approximation)

$$\underbrace{\sqrt{2\pi}}_{= 2.5066} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \leq n! \leq \underbrace{e}_{= 2.7183} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}$$

Huge: Grows exponentially in n

Distinct Letters

"How many sequences of 5 ***distinct*** *alphabet letters from* $\{A, B, \ldots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26 \times 25 \times 24 \times 23 \times 22 = 7893600$

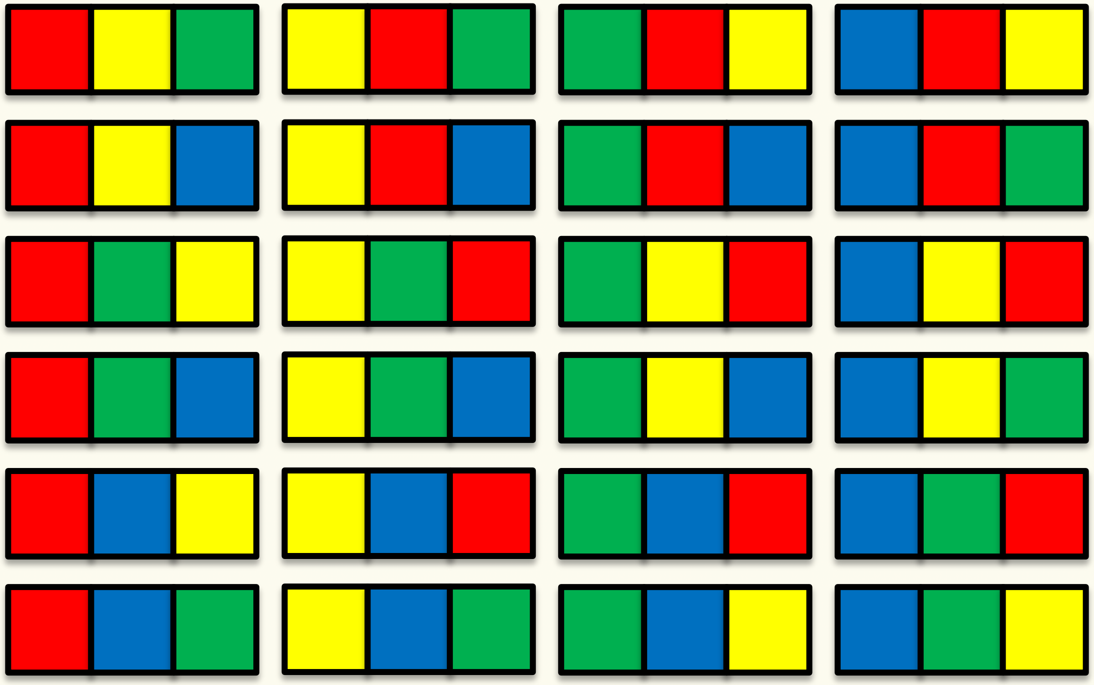
In general

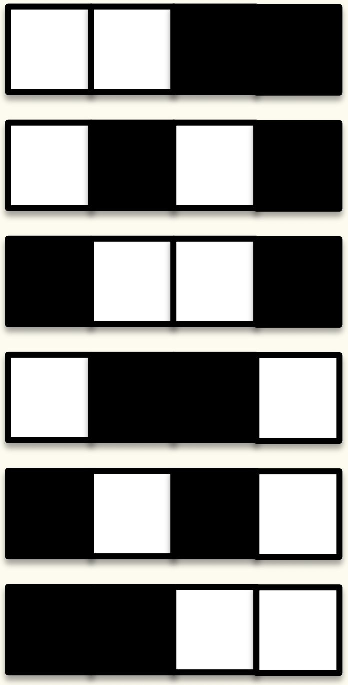
Fact. # of k-element sequences of distinct symbols from an n-element set is

$$P(n,k) = n\times(n-1)\times\cdots\times(n-k+1) = \frac{n!}{(n-k)!}$$

Today: More Counting

- Permutations and Combinations





Number of Subsets

"How many size-5 ***subsets*** *of* $\{A, B, \ldots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not:
{S,T,E,V}, {S,A,R,H},…

Difference from k-permutations: NO ORDER

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

Number of Subsets – Idea

Consider a sequential process:

1. Choose a subset $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: A sequence of 5 distinct letters from $\{A, B, \dots, Z\}$

$$\binom{26}{5} \times 5! = \frac{26!}{21!}$$

$$\binom{26}{5} = \frac{26!}{21!\,5!} = 65780$$

Number of Subsets – Binomial Coefficient

Fact. The number of subsets of size k of a set of size n is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof, i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!k!} = \binom{n}{n-k}$

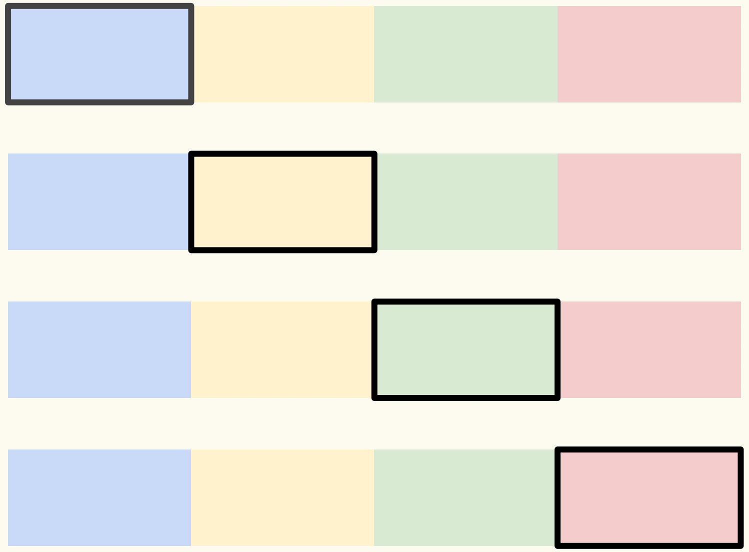
Why??

Symmetry in Binomial Coefficients – A different proof

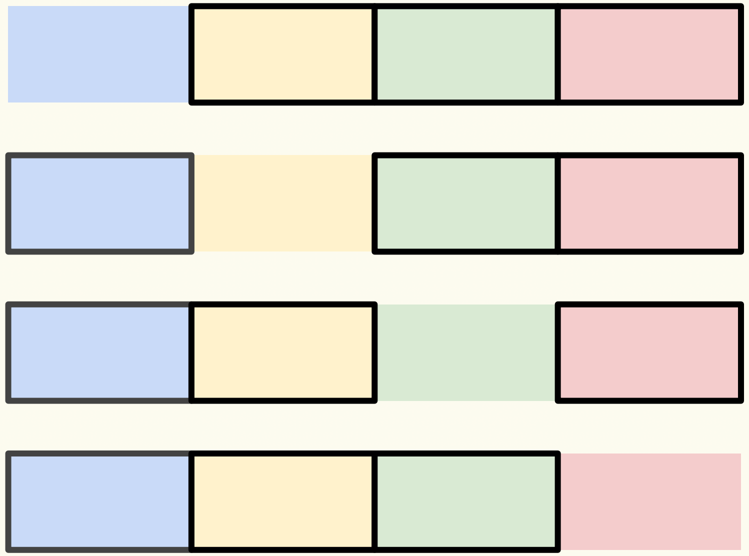
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

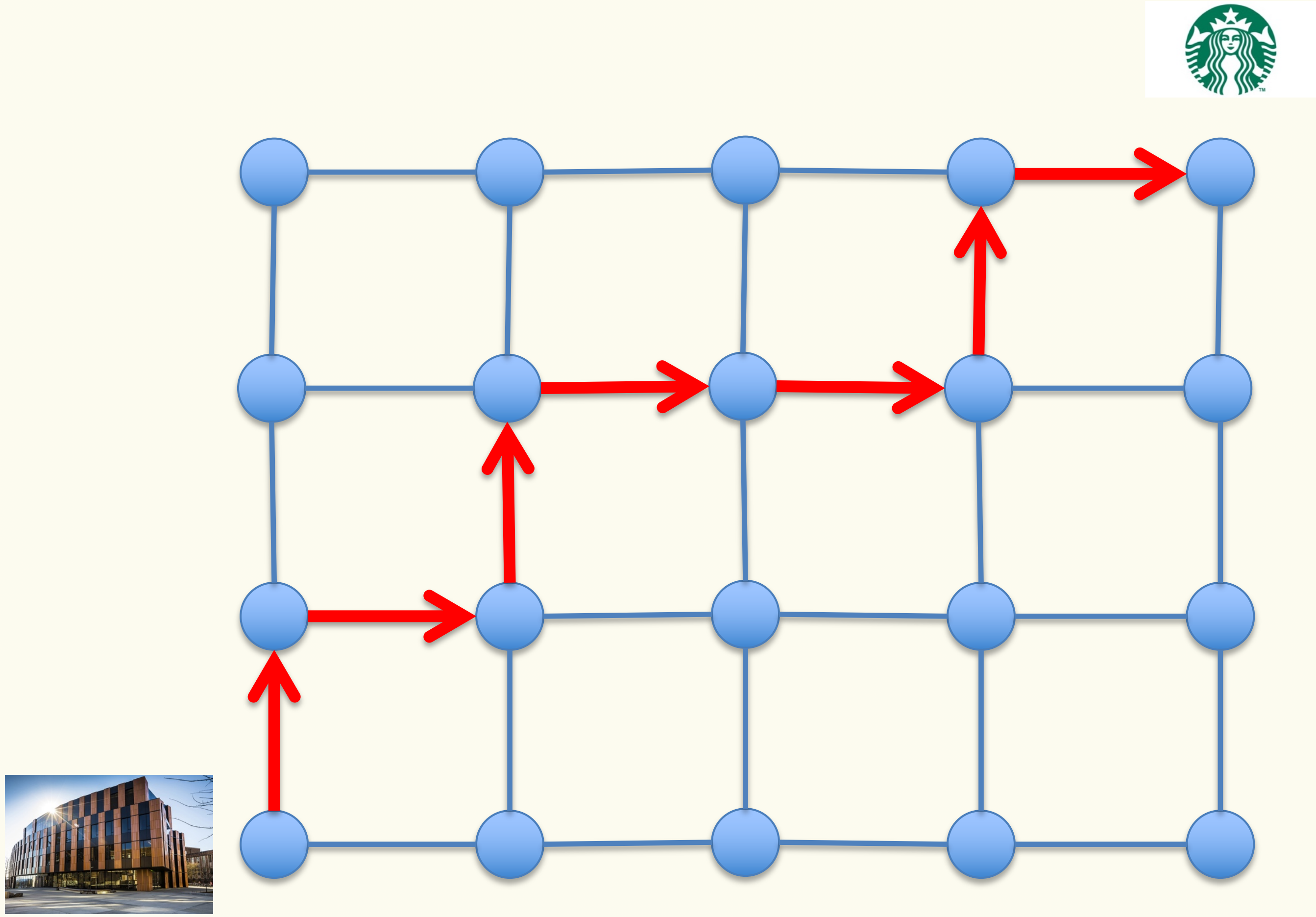
1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$

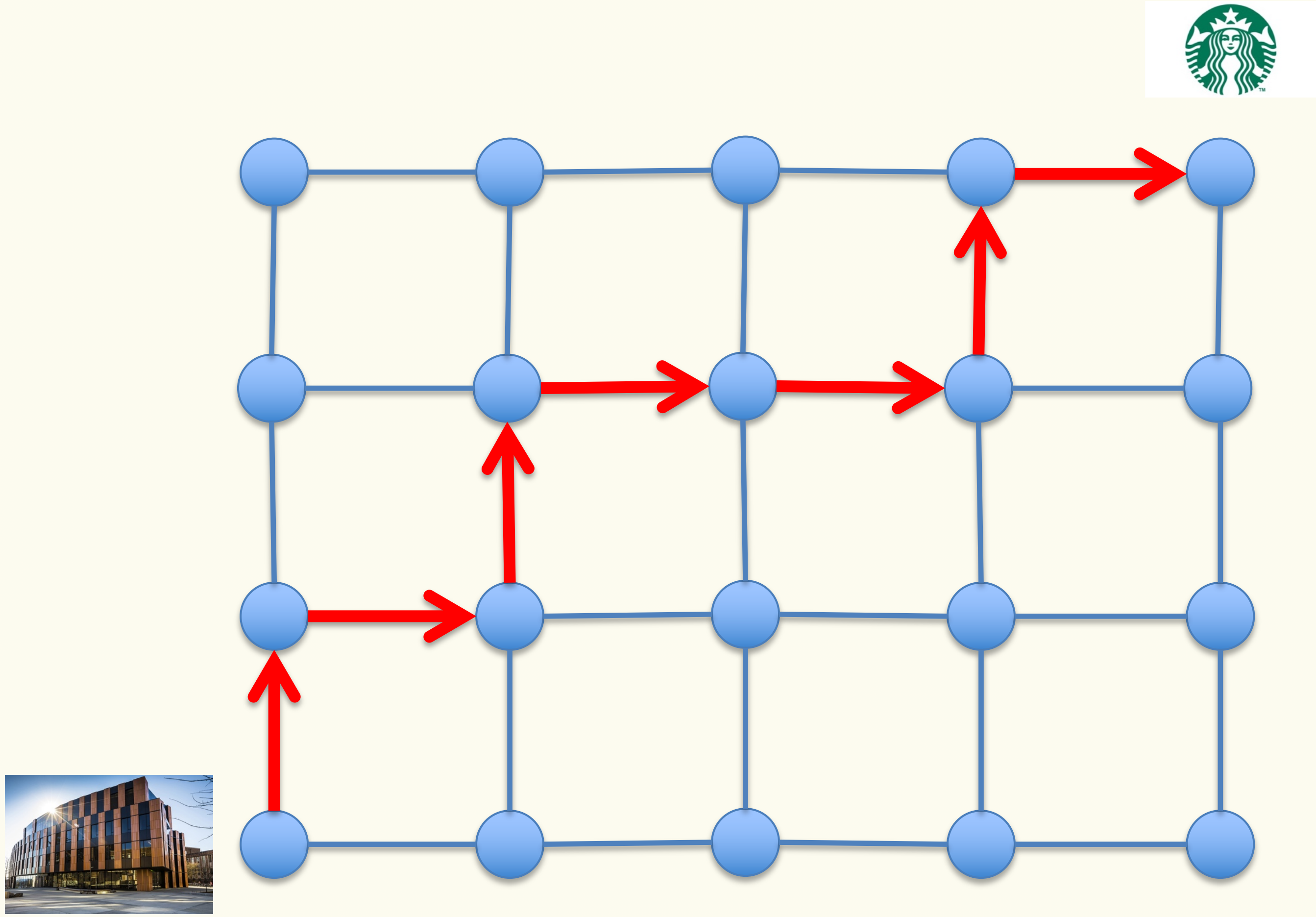


Example – Counting Paths



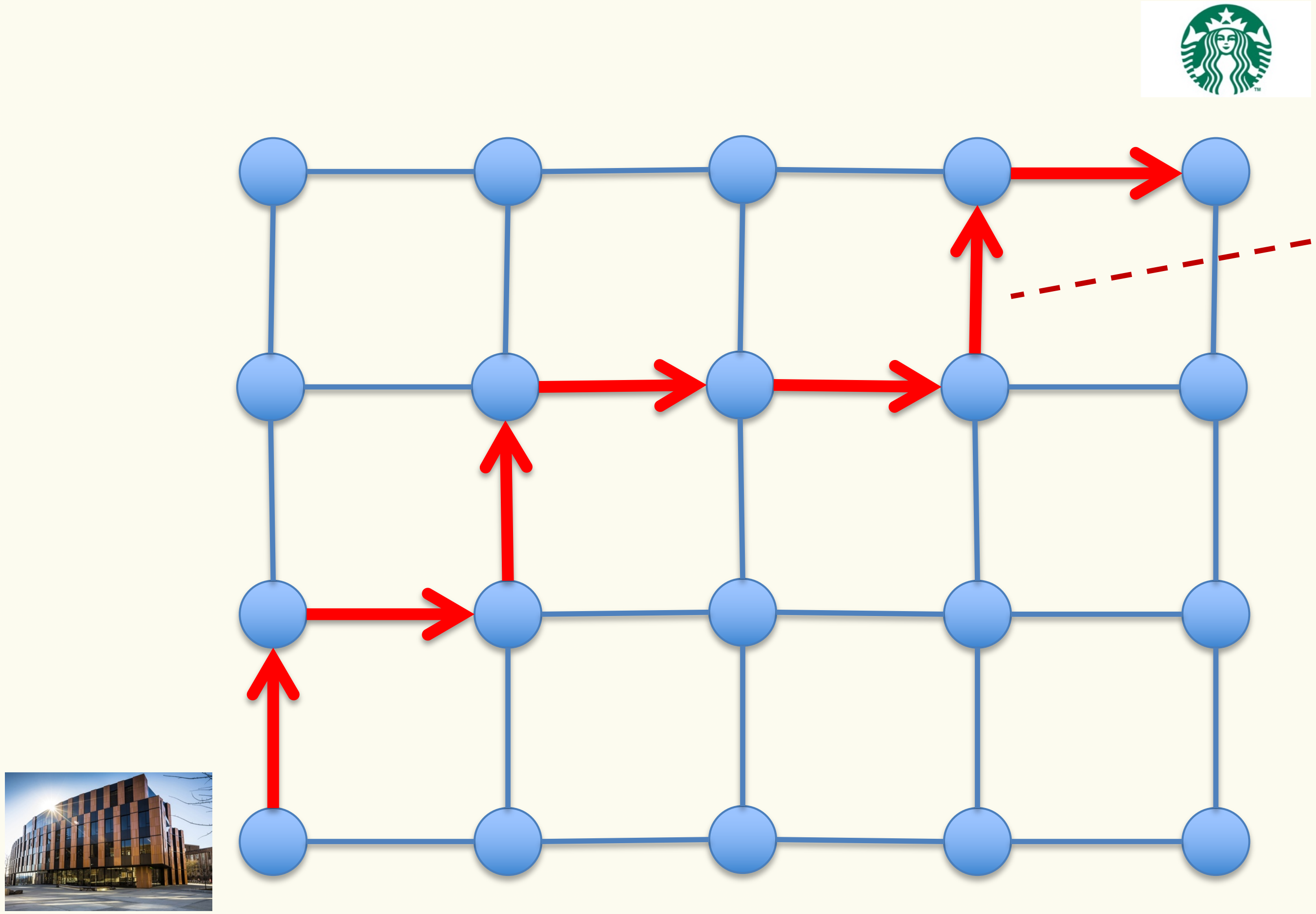
"How many shortest paths from Gates to Starbucks?"

Example – Counting Paths



How do we represent a shortest path?

Example – Counting Paths



$$\text{Path} \in \{\uparrow, \rightarrow\}^7$$

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

$\uparrow$'s =3, # $\rightarrow$'s=4

Poll: # of shortest paths?

A. 2^7

B. $\frac{7!}{4!}$

C. $\binom{7}{4} = \frac{7!}{4!3!}$

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3, n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Hint: we can represent each solution as a binary string.

"*How many solutions* $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?*"

Example: $k = 3, n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

Clever representation of solutions

$(3,1,1)$	$(2,1,2)$	$(1,0,4)$
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1

"How many solutions $(x_1, \dots, x_k)$ such that $x_1, \dots, x_k \geq 0$ and $\sum_{i=1}^{k} x_i = n$?"

Example: $k = 3, n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$	$(2,1,2)$	$(1,0,4)$
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1

Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

The problem reduces to counting combinations!

*aka. "stars and bars method"

More general counting using binary encoding*

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Example with k non-negative integers summing to n:
bins are the k integers, balls are the n 1's that add to n.

Coins

How many ways can you distribute 32 identical coins among Alex, Barbara, Charlie, Dana, and Eve?

1. Identify balls
2. Identify bins

$$\binom{32+5-1}{5-1}$$

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Guess: 7!   Correct?!

No! e.g., swapping two T's also leads to *SEATTLE*
swapping two E's also leads to *SEATTLE*

Counted as separate permutations, but they lead to the same word.

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1$$

Locations of T's

Locations of E's

Location of S

Location of A

Location of L

T E S T A L E

Another way to look at SEATTLE

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2!\,\cancel{5}!} \times \frac{\cancel{5}!}{2!\,\cancel{3}!} \times \cancel{3}!$$

$$= \frac{7!}{2!\,2!} = 1260$$

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T's, and divide by 2! again for 2 duplicate E's.

More generally…

How many ways can you arrange the letters in "Godoggy"?

$n = 7$ Letters, $k = 4$ Types {G, O, D, Y}

$n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 1$

$$\frac{7!}{3!2!1!1!} = \binom{7}{3,2,1,1}$$

Multinomial coefficients

Multinomial Coefficients

If we have k types of objects ($\boldsymbol{n}$ total), with $\boldsymbol{n_1}$ of the first type, $\boldsymbol{n_2}$ of the second, …, and $\boldsymbol{n_k}$ of the k^{th}, then the number of orderings possible is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Binomial Coefficients – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Symmetry in Binomial Coefficients

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Binomial Theorem

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Identity
(Next lecture)

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

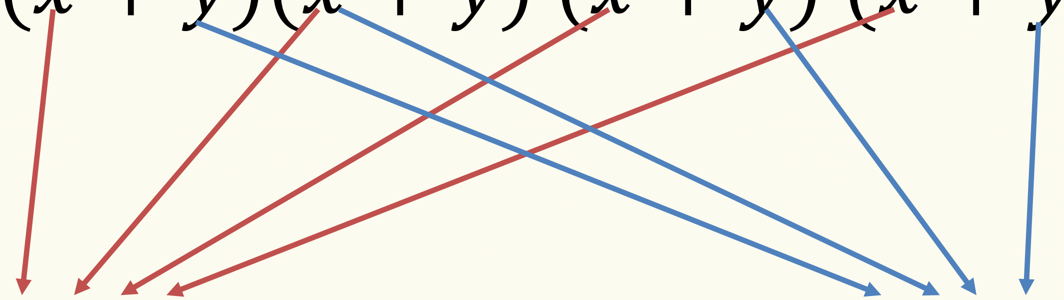
C. $\binom{4}{3}$

D. 3

https://pollev.com/paulbeame028

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y)\\ &= xx+xy+yx+yy\\ &= x^2+2xy+y^2\end{aligned}$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x + y)^n = (x + y) \dots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x + y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x + y)$ (the other $n - k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Apply with $x = y = 1$

Many properties of sums of binomial coefficients can be found by plugging in different values of x and y in the Binomial Theorem.

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Quick Summary

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule → n^k
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation → $\frac{n!}{(n-k)!}$
- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination → $\binom{n}{k} = \frac{n!}{k!(n-k)!}$