Quiz Section 9

Review

1) Maximum Likelihood Estimator (MLE): We denote the MLE of θ as $\hat{\theta}_{\text{MLE}}$ or simply $\hat{\theta}$, the parameter (or vector of parameters) that maximizes the likelihood function (probability of seeing the data).

$$\hat{\theta}_{\text{MLE}} = \arg\max_{\theta} \mathcal{L}(x_1, \ldots, x_n \mid \theta) = \arg\max_{\theta} \ln \mathcal{L}(x_1, \ldots, x_n \mid \theta)$$

2) An estimator $\hat{\theta}$ for a parameter θ of a probability distribution is **unbiased** iff $\mathbb{E}[\hat{\theta}(X_1, \ldots, X_n)] = \theta$

Task 1 – Mystery Dish!

A fancy new restaurant has opened up which features only 4 dishes. The unique feature of dining here is that they will serve you any of the four dishes randomly according to the following probability distribution: give dish A with probability 0.5, dish B with probability θ, dish C with probability 2θ, and dish D with probability $0.5 - 3\theta$. Each diner is served a dish independently. Let x_A be the number of people who received dish A, x_B the number of people who received dish B, etc, where $x_A + x_B + x_C + x_D = n$. Find the MLE for θ, $\hat{\theta}$.

Task 2 – A Red Poisson

Suppose that $x_1, \ldots, x_n$ are i.i.d. samples from a Poisson(θ) random variable, where θ is unknown. In other words, they follow the distributions $\mathbb{P}(k; \theta) = \theta^k e^{-\theta}/k!$, where $k \in \mathbb{N}$ and $\theta > 0$ is a positive real number.

Find the MLE of θ.

Task 3 – A biased estimator

In class, we showed that the maximum likelihood estimate of the variance θ_2 of a normal distribution (when both the true mean μ and true variance σ^2 are unknown) is what's called the *population variance*. That is

$$\hat{\theta}_2 = \left(\frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\theta}_1)^2 \right)$$

where $\hat{\theta}_1 = \frac{1}{n} \sum_{i=1}^{n} x_i$ is the MLE of the mean. Is $\hat{\theta}_2$ unbiased?

Task 4 – Weather Forecast

A weather forecaster predicts sun with probability θ_1, clouds with probability $\theta_2 - \theta_1$, rain with probability $\frac{1}{2}$ and snow with probability $\frac{1}{2} - \theta_2$. This year, there have been 55 sunny days, 100 cloudy days, 160 rainy days and 50 snowy days. What is the maximum likelihood estimator for θ_1 and θ_2?

Task 5 – Pareto

The Pareto distribution was discovered by Vilfredo Pareto and is used in a wide array of fields but particularly social sciences and economics. It is a density function with a slowly decaying tail, for example it can describe the wealth distribution (a small group at the top holds most of the wealth). We consider its special form given by the family of Pareto distributions Pareto$(1, \alpha)$ with densities[1]

$$f(x; \alpha) = \frac{\alpha}{x^{\alpha+1}}$$

where $x \geqslant 1$ and the real number $\alpha \geqslant 0$ is the parameter. Moreover, $f(x; \alpha) = 0$ for $x < 1$. You are given i.i.d. samples $x_1, x_2, \ldots, x_n$ from the Pareto distribution with parameter α. Find the MLE estimation of α.

[1] The more general Pareto distribution depends on an additional real positive parameter m and follows the density $f(x; \alpha, m) = \frac{\alpha \cdot m^\alpha}{x^{\alpha+1}}$ for $x \geqslant m$, and is 0 for $x < m$. Here, we consider the special case with $m = 1$.