Quiz Section 4 – Solutions

Review

1) Probability Mass. For every random variable X, we have $\sum_x \mathbb{P}(X = x) =$ ______.

2) Expectation. $\mathbb{E}[X] =$ ______.

3) Linearity of expectation. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = \rule{4cm}{0.4pt}\ .$$

4) Variance. $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[\mathbb{X}])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \qquad \mathrm{Var}(aX + b) = \underline{\quad}\mathrm{Var}(X)$.

5) Independence. Two random variables X and Y are **independent** if ______________.

6) Variance and Independence. For any two independent random variables X and Y, $\mathrm{Var}(X + Y) =$ ______________

Task 1 – Identify that range!

Identify the support/range Ω_X of the random variable X, if X is...

a) The sum of two rolls of a six-sided die.

X takes on every integer value between the min sum 2, and the max sum 12.
$\Omega_X = \{2, 3, ..., 12\}$

b) The number of lottery tickets I buy until I win it.

X takes on all positive integer values (I may never win the lottery).
$\Omega_X = \{1, 2, ...\} = \mathbb{N}$

c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

X takes on every integer value between the min number of heads 0, and the max n.
$\Omega_X = \{0, 1, ..., n\}$

d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Since $\mathbb{P}(head) = 1$, we are guaranteed to get n heads in n flips.
$\Omega_X = \{n\}$

Task 2 – Symmetric Difference

Suppose A and B are random, independent (possibly empty) subsets of $\{1, 2, \ldots, n\}$, where each subset is equally likely to be chosen as A or B. Consider $A \Delta B = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (A^C \cup B^C)$, i.e., the set containing elements that are in exactly one of A and B. Let X be the random variable that is the size of $A \Delta B$. What is $\mathbb{E}[X]$?

For $i = 1, 2, \ldots, n$, let X_i be the indicator of whether $i \in A \Delta B$. Further, let Y_i and Z_i be the indicator variables of whether $i \in A$ and $i \in B$, respectively. Then,

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(Z_i = 1) = \frac{2^{n-1}}{2^n} = \frac{1}{2},$$

where we have used the fact that exactly half of the 2^n subsets of $[n]$ contain i. Further,

$$\begin{aligned}\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) &= \mathbb{P}(Y_i = 0, Z_i = 1) + \mathbb{P}(Y_i = 1, Z_i = 0) \\ &= \mathbb{P}(Y_i = 0)\,\mathbb{P}(Z_i = 1) + \mathbb{P}(Y_i = 1)\,\mathbb{P}(Z_i = 0) \\ &= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2},\end{aligned}$$

where we have used the fact that Y_i, Z_i are independent.

Then $X = \sum_{i=1}^{n} X_i$, so

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \frac{n}{2}$$

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Task 3 – Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

The washing machine eats 4 socks every time. It can either eat a single sock from 4 pairs of socks, leaving us with 6 complete pairs, or a single sock from 2 pairs and a matching pair, leaving us with 7 complete pairs, or 2 pairs of matching socks, leaving us with 8 complete pairs.

$$\Omega_X = \{6, 7, 8\}$$

We are dealing with a sample space with equally likely outcomes. As such, we can compute use the formula $P(E) = \frac{|E|}{|\Omega|}$. We know that $|\Omega| = \binom{20}{4}$ because the washing machine picks a set of 4 socks out of 20 possible socks.

To define the pmf of X, we consider each value in the range of X.

For $k = 6$, we first pick 4 out of 10 pairs of socks from which we will eat a single sock ($\binom{10}{4}$ ways), and for each of these 4 pairs we have two socks to pick from ($\binom{2}{1}^4$ ways). Using the product rule, we get $|X = 6| = \binom{10}{4}2^4$.

For $k = 7$, we first pick 1 out of 10 pairs of socks to eat in its entirety ($\binom{10}{1}$ ways), and then 2 out of the 9 remaining pairs from which we will eat a single sock ($\binom{9}{2}$ ways), and for each of these 2 pairs we have two socks to pick from ($\binom{2}{1}^2$ ways). Using the product rule, we get $|X = 7| = 10\binom{9}{2}2^2$.

For $k = 8$, we pick 2 out of 10 pairs of socks to eat ($\binom{10}{2}$ ways). We get $|X = 8| = \binom{10}{2}$.

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{2}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \end{cases}$$

b) Find $\mathbb{E}[X]$ from the definition of expectation.

$$\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k) = 6 \cdot \frac{\binom{10}{4}2^4}{\binom{20}{4}} + 7 \cdot \frac{10\binom{9}{2}2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \boxed{\frac{120}{19}}$$

c) Find $\mathbb{E}[X]$ using linearity of expectation.

For $i \in [10]$, let X_i be 1 if pair i survived, and 0 otherwise. Then, $X = \sum_{i=1}^{10} X_i$. But $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{\binom{18}{4}}{\binom{20}{4}}$, where the numerator indicates the number of ways of choosing 4 out the 18 remaining socks (we spare our chosen pair i). Hence,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{10} X_i] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i=1}^{10} \frac{\binom{18}{4}}{\binom{20}{4}} = 10\frac{\binom{18}{4}}{\binom{20}{4}} = \boxed{\frac{120}{19}}$$

d) Which way was easier? Doing both (a) and (b), or just (c)?

Part (c) is was probably much easier. In this problem, you may have found part (a) and (b) easier, because there were only 3 possible values in the range of X. However, in general computing the probability mass function of complicated random variables (ones with hundreds of elements in their range) can be very difficult. Often it is much easier to use linearity of expectation and compute the probability mass function of simpler random variables.

Task 4 – Balls in Bins

Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$.

For $i \in [n]$, let X_i be 1 if bin i is empty, and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$. We first compute $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = (\frac{n-1}{n})^m$. Indeed, we are assuming multiple balls can go in the same bin. As such, when computing $P(X_i = 1)$, given that bin i is empty, we remove it from the pool of possible bins to pick from, leaving us with $n - 1$ bins out of a total of n bins in which we can place balls. Since we are distributing m balls over the n bins, the event that bin i remains empty occurs with probability $\left(\frac{n-1}{n}\right)^m$. Hence, by linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot \left(\frac{n-1}{n}\right)^m$$

Task 5 – Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

a) Find $p_X(k)$, the probability mass function for X.

Let L be a left step, R be a right step, and N be no step.
The range of X is $\{-2, -1, 0, 1, 2\}$. We can compute $p_X(-2) = \mathbb{P}(X = -2) = \mathbb{P}(LL) = p_2^2$, $p_X(-1) = \mathbb{P}(X = -1) = \mathbb{P}(LN \cup NL) = 2p_2p_3$, and $p_X(0) = \mathbb{P}(X = 0) = \mathbb{P}(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$. Similarly for $p_X(1)$ and $p_X(2)$.

$$p_X(k) = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \end{cases}$$

b) Compute $\mathbb{E}[X]$ from the definition.

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$

c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation.

Let Y be the amount you moved on the first step (either $-1, 0, 1$), and Z the amount you moved on the second step. Then, $\mathbb{E}[Y] = \mathbb{E}[Z] = (1)(p_1) + (0)(p_3) + (-1)(p_2) = p_1 - p_2$.
Then $X = Y + Z$ and $\mathbb{E}[X] = \mathbb{E}[Y + Z] = \mathbb{E}[Y] + \mathbb{E}[Z] = 2(p_1 - p_2)$

Task 6 – 3-sided Die

Let the random variable X be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

a) What is the probability mass function of X?

First let us define the range of X. A three sided-die can take on values $1, 2, 3$. Since X is the sum of two rolls, the range of X is $\Omega_X = \{2, 3, 4, 5, 6\}$.

We can then define the pmf of X. To that end, we must define two random variables R_1, R_2 with R_1 being the roll of the first die, and R_2 being the roll of the second die. Then, $X = R_1 + R_2$. Note that $\Omega_{R1} = \Omega_{R2} = \{1, 2, 3\}$. With that in mind we can find the pmf of X:

$$\begin{aligned} p_X(k) = \mathbb{P}(X = k) &= \sum_{i \in \Omega_{R1}} \mathbb{P}(R_1 = i, R_2 = k - i) \\ &= \sum_{i \in \Omega_{R1}} \mathbb{P}(R_1 = i) \cdot \mathbb{P}(R_2 = k - i) \qquad \text{(By independence of the rolls)} \\ &= \sum_{i \in \Omega_{R1}} \frac{1}{3} \cdot p_{R2}(k - i) \\ &= \frac{1}{3}\left(p_{R2}(k-1) + p_{R2}(k-2) + p_{R2}(k-3)\right) \end{aligned}$$

At this point, we can evaluate the pmf of X for each value in the range of X, noting that $p_{R2}(k-i) = 0$ if $k - i \notin \Omega_{R2}$, $1/3$ otherwise. We get:

$$p_X(k) = \begin{cases} 1/9 & k = 2 \\ 2/9 & k = 3 \\ 3/9 & k = 4 \\ 2/9 & k = 5 \\ 1/9 & k = 6 \end{cases}$$

One could also list out the possible values of the first two rolls and use a table to find the marginal pmf of X by summing up the entries of each row for each $k \in \Omega_X$.

b) Find $\mathbb{E}[X]$ directly from the definition of expectation.

$$\mathbb{E}[X] = \sum_{k=2}^{6} k p_X(k) = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = \boxed{4}$$

c) Find $\mathbb{E}[X]$ again, but this time using linearity of expectation.

Let R_1 be the roll of the first die, and R_2 the roll of the second. Then, $X = R_1 + R_2$.
By linearity of expectation, we get:

$$\mathbb{E}[X] = \mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$$

We compute:

$$\mathbb{E}[R_1] = \sum_{i \in \Omega_{R1}} i \cdot \mathbb{P}(R_1 = i) = \sum_{i \in \Omega_{R1}} i \cdot \frac{1}{3} = \frac{1}{3}(1 + 2 + 3) = 2$$

Similarly, $\mathbb{E}[R_2] = 2$, since the rolls are independent.

Plugging into our expression for the expectation of X gives us:

$$\mathbb{E}[X] = 2 + 2 = \boxed{4}$$

d) What is $\mathrm{Var}(X)$?

We know from the definition of variance that

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

We can compute the $\mathbb{E}[X^2]$ term as follows:

$$\mathbb{E}[X^2] = \sum_{x=2}^{6} x^2 p_X(x) = \frac{2^2 \cdot 1 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 2 + 6^2 \cdot 1}{9} = \frac{52}{3}$$

Plugging this into our variance equation gives us

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{52}{3} - 4^2 = \boxed{\frac{4}{3}}$$

Task 7 – Practice

a) Let X be a random variable with $p_X(k) = ck$ for $k \in \{1, \ldots, 5\} = \Omega_X$, and 0 otherwise. Find the value of c that makes X follow a valid probability distribution and compute its mean and variance ($\mathbb{E}[X]$ and $\mathrm{Var}(X)$).

For X to follow a valid probability distribution, we must have $\sum_{k \in \Omega_X} p_X(k) = 1$. We can solve for c so that the equality holds. We know:

$$\sum_{k \in \Omega_X} p_X(k) = \sum_{k \in \Omega_X} ck = c \sum_{k \in \Omega_X} k) = c \cdot (1 + 2 + 3 + 4 + 5) = 15c$$

So for the normalization of the pmf of X to hold, we must choose $c = 1/15$.
We can now use the definition of expectation:

$$\mathbb{E}[X] = 1 \cdot \frac{1}{15} + 2 \cdot \frac{2}{15} + 3 \cdot \frac{3}{15} + 4 \cdot \frac{4}{15} + 5 \cdot \frac{5}{15} = 55/15 \approx \boxed{3.667}$$

And compute $E[X]$ as follows:

$$\mathbb{E}[X^2] = 1^2 \cdot \frac{1}{15} + 2^2 \cdot \frac{2}{15} + 3^2 \cdot \frac{3}{15} + 4^2 \cdot \frac{4}{15} + 5^2 \cdot \frac{5}{15} = 225/15 = \boxed{15}$$

And the variance of X:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X] = 15 - (55/15)^2 = \frac{15^3 - 55^2}{15} = \frac{350}{225} = \frac{14}{9} \approx \boxed{1.556}$$

b) Let X be *any* random variable with mean $\mathbb{E}[X] = \mu$ and variance $\mathrm{Var}(X) = \sigma^2$. Find the mean and variance of $Z = \dfrac{X - \mu}{\sigma}$. (When you're done, you'll see why we call this a "standardized" version of X!)

We know that $\mathbb{E}[aX] = a \cdot \mathbb{E}[X]$ for some constant a, and that $\mathbb{E}[X + b] = \mathbb{E}[X] + b$ for some constant b. As such, we can compute the expectation of the standardized version of X, knowing that $E[X] = \mu$:

$$\mathbb{E}[Z] = \mathbb{E}\left[\frac{X - \mu}{\sigma}\right] = \frac{1}{\sigma}\left(\mathbb{E}[X - \mu]\right) = \frac{1}{\sigma}(\mathbb{E}[X] - \mu) = \boxed{0}$$

For the variance, we know that $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$. With that in mind, knowing that $\mathrm{Var}(X) = \sigma^2$, we can write:

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{X - \mu}{\sigma}\right) = \frac{1}{\sigma^2}\mathrm{Var}(X) = \boxed{1}$$

c) Let X, Y be independent random variables. Find the mean and variance of $X - 3Y - 5$ in terms of $\mathbb{E}[X], \mathbb{E}[Y], \mathrm{Var}(X)$, and $\mathrm{Var}(Y)$.

Using the linearity of expectation, we can write:

$$\mathbb{E}[X - 3Y - 5] = \mathbb{E}[X] - 3\mathbb{E}[Y] - 5$$

We also know that the variance of a sum of independent random variables A and B is the sum of their variances, so that $\mathrm{Var}(A + B) = \mathrm{Var}(A) + \mathrm{Var}(B)$. In our case, we have $A = X$, and $B = -3Y$. We get:

$$\mathrm{Var}(X - 3Y - 5) = \mathrm{Var}(X) + \mathrm{Var}(-3Y) = \mathrm{Var}(X) + 9\mathrm{Var}(Y)$$

d) Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables each with mean μ and variance σ^2. The sample mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Find the mean and variance of $\bar{X}$. If you use the independence assumption anywhere, **explicitly label** at which step(s) it is necessary for your equalities to be true.

Using linearity of expectation,

$$\mathbb{E}[\overline{X}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{n}n\mu = \mu$$

$$\mathrm{Var}\left(\overline{X}\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left(X_i\right) = \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}$$

In the calculation for the variance, we used the independence of the X_i's.

Task 8 – Expectations, Independence, and Variance

a) Let U be a random variable which is uniform over the set $[n] = \{1, 2, \ldots, n\}$, i.e, $\mathbb{P}(U = i) = \frac{1}{n}$ for all $i \in [n]$. Compute $\mathbb{E}\left[U^2\right]$ and $\mathrm{Var}(U)$.

Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

First off, note that

$$\mathbb{E}\left[U^2\right] = \frac{1}{n}\sum_{i=1}^{n} i^2 = \frac{1}{n}\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

by the hint. Also, note that

$$\mathbb{E}\left[U\right] = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}\ .$$

Therefore

$$\begin{aligned}\mathrm{Var}(U) &= \mathbb{E}\left[U^2\right] - \mathbb{E}\left[U\right]^2 \\ &= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} \\ &= \frac{n+1}{12}\cdot(4n+2-3n-3) = \frac{(n+1)(n-1)}{12}\ .\end{aligned}$$

b) Let Y_1 and Y_2 be the independent outcomes of two dice rolls, and let $Z = Y_1 + Y_2$. Then, compute $\mathbb{E}\left[Z^2\right]$ and $\mathrm{Var}(Z)$.

Hint: Try to use an indirect solution using linearity and independence, without the need of explicitly giving the distribution of Z^2.

First note that by linearity and independence,

$$\mathbb{E}\left[Z^2\right] = \mathbb{E}\left[Y_1^2\right] + \mathbb{E}\left[Y_2^2\right] + 2\mathbb{E}\left[Y_1 \cdot Y_2\right] = \mathbb{E}\left[Y_1^2\right] + \mathbb{E}\left[Y_2^2\right] + 2\mathbb{E}\left[Y_1\right]\mathbb{E}\left[Y_2\right]\ .$$

We know that $\mathbb{E}\left[Y_1\right] = \mathbb{E}\left[Y_2\right] = 21/6$. We also know that $\mathbb{E}\left[Y_1^2\right] = \mathbb{E}\left[Y_2^2\right] = 91/6$ (from class). Thus,

$$\mathbb{E}\left[Z^2\right] = 91/3 + 2\cdot 21^2/36 = 91/3 + 147/6 = 329/6\ .$$

On the other hand, we know that $\mathbb{E}[Z] = 7$. Therefore,

$$\mathrm{Var}(Z) = \mathbb{E}\left[Z^2\right] - \mathbb{E}[Z]^2 = 329/6 - 294/6 = 35/6\ .$$

We could also have used $\mathrm{Var}(Z) = \mathrm{Var}(Y_1 + Y_2) = \mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) = 35/12 \cdot 2 = 35/6$, using the calculation from class for the individual variances.