CSE 312

Foundations of Computing II

Linearity of expectation, LOTUS

Lecture 9: ~~Variance and Independence of RVs~~

Anonymous questions:

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Agenda

- Recap
- Linearity of expectation
- LOTUS
- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Review Random Variables

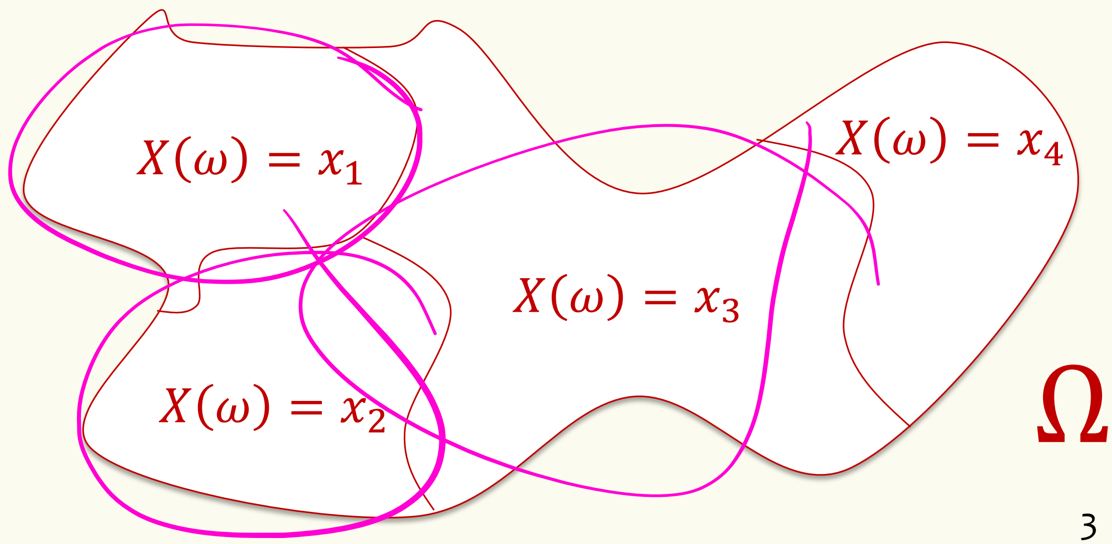
Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is its *range/support*: $\boldsymbol{X(\Omega)}$

$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables **partition** the sample space.

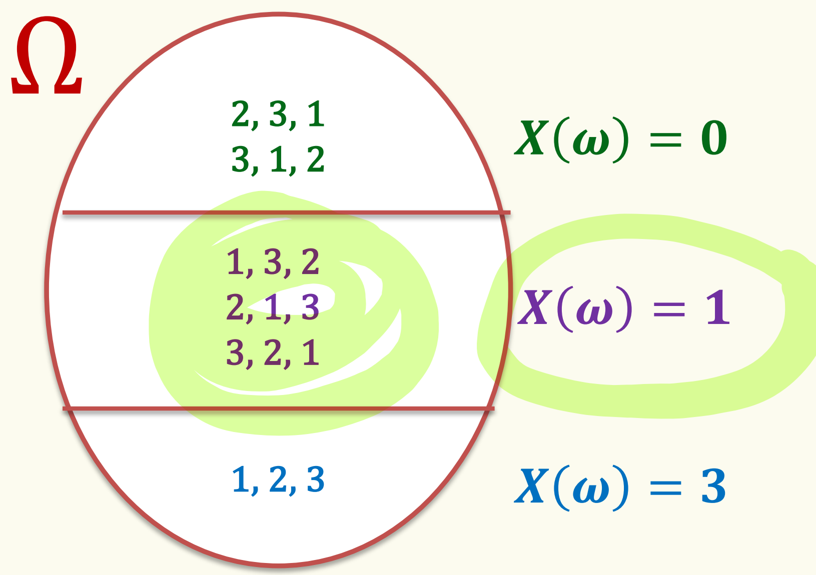
$\Sigma_{x \in X(\Omega)} P(X = x) = 1$



Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1



Review Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Example: Returning Homeworks

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\mathbf{Pr}(\boldsymbol{\omega})$	$\boldsymbol{\omega}$	$\boldsymbol{X}(\boldsymbol{\omega})$
$1/6$	$1, 2, 3$	3
$1/6$	$1, 3, 2$	1
$1/6$	$2, 1, 3$	1
$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= 3 \cdot \frac{1}{6} + 1 \cdot \frac{3}{6} + 0 \cdot \frac{2}{6}$$

$$= 3 \cdot P(X = 3) + 1 \cdot P(X = 1) + 0 \cdot P(X = 0)$$

$$= 1$$

$E(X_1+X_2+\dots+X_n) = E(X_1)+E(X_2)+\dots+E(X_n)$

$= \sum_{i=1}^{n} E(X_i)$

Recap Linearity of Expectation

Theorem. For any two random variables X and Y *(X, Y do not need to be independent)*

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$a=3$
$b=17$

Theorem. For any random variables X, and constants a and b

$$\mathbb{E}[aX+b] = a \cdot \mathbb{E}[X] + b.$$

$E(\underline{3X+17})$
$= 3E(X) + 17$

For any event A, can define the indicator random variable X for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$P(X_A = 1) = P(A)$
$P(X_A = 0) = 1 - P(A)$

$E(X_A) = 1 \cdot P(A) + 0 \cdot P(\bar{A}) = P(A)$

X_i : $\{A$: i^{th} coin flip $H\}$

$E(X_i) = P(i^{th}$ flip is $H) = p$

Example – Coin Tosses – The brute force method

We flip n coins, each one heads with probability p,

Z is the number of heads, what is $\mathbb{E}[Z]$?

$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z=k) \quad = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^{k} (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k} (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$

Can we solve it more elegantly, please?

X : # heads in n indep tosses
p of H

Computing complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

Often, X_i are indicator (0/1) random variables.

Example – Coin Tosses

coin flips

We flip n ~~coins~~, each toss independent, comes up heads with probability p

Z is the number of heads, what is $\mathbb{E}[Z]$?

$$X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$$

Fact. $Z = X_1 + \cdots + X_n$

Outcomes	X_1	X_2	X_3	Z
TTT	0	0	0	0
TTH	0	0	1	1
THT	0	1	0	1
THH	0	1	1	2
HTT	1	0	0	1
HTH	**1**	**0**	**1**	**2**
HHT	1	1	0	2
HHH	1	1	1	3

$n = 3$

$E(Z) = E(X_1) + E(X_2) + \dots + E(X_n)$

$E(X_i) = P(i^{\text{th}} \text{ flip is } H) = p$

$E(Z) = np$

Example – Coin Tosses

We flip n coins, each toss independent, comes up heads with probability p

Z is the number of heads, what is $\mathbb{E}[Z]$?

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$

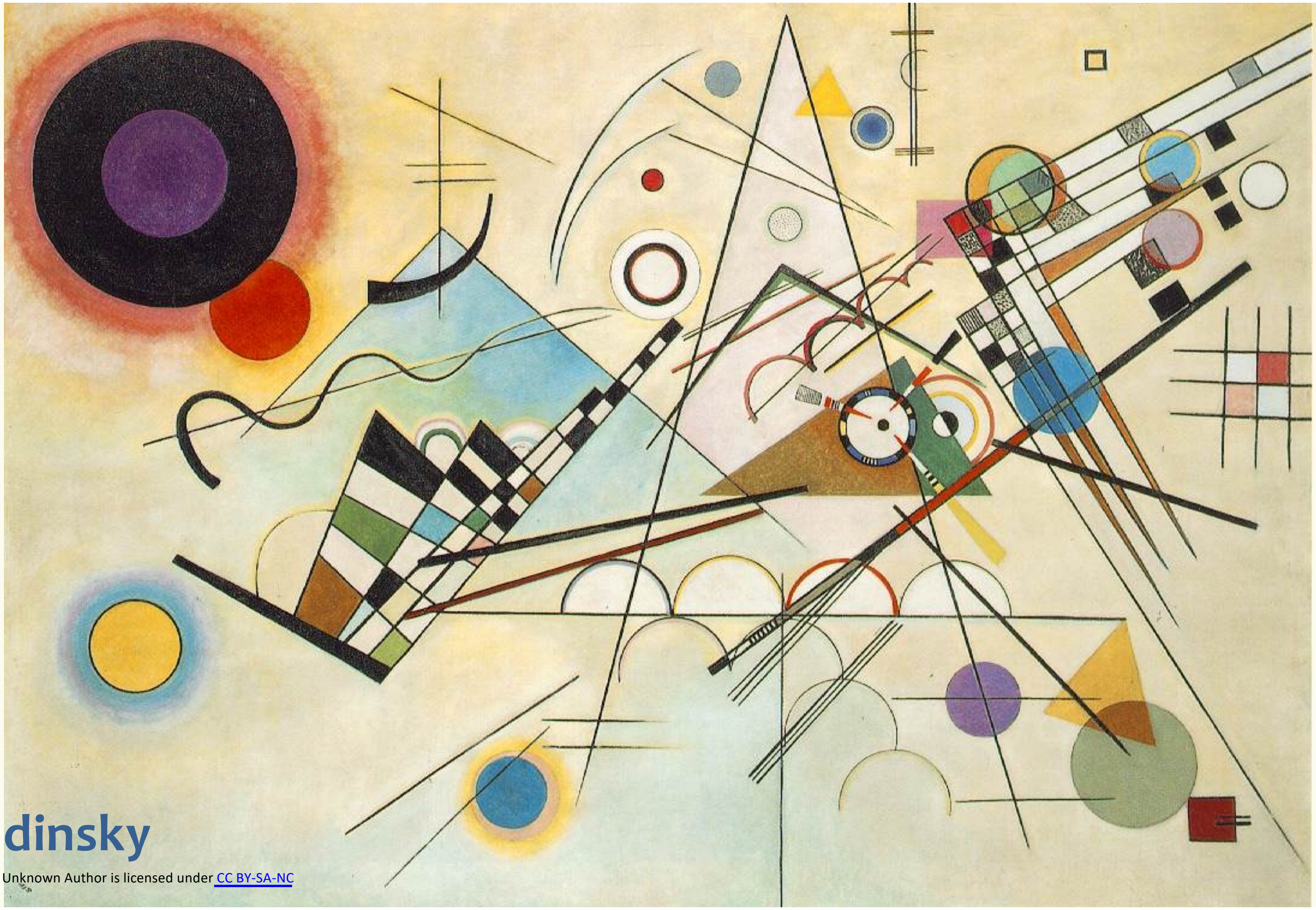
Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \quad = n \cdot p$$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

$$\mathbb{E}[X_i] = p \cdot 1 + (1-p) \cdot 0 = p$$



Kandinsky

$\Pr(E) = \frac{|E|}{n!}$ ↑ 1, M $\frac{(n-1)!}{n!} = \frac{1}{n}$

1 i n

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$?

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$X_i = \begin{cases} 1 & \text{if person } i \text{ got their own hw back} \\ 0 & \text{o.w.} \end{cases}$

$X = \sum_{i=1}^{n} X_i$

$\underset{\text{LOE}}{\Rightarrow}$ $E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n}$

$E(X_i) = P(\text{person } i \text{ got their own hw back})$

$= \frac{1}{n}$ $= 1$

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
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Decompose: Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Conquer: Compute the expectation of each X_i and sum!

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
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$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1

Decompose: What is X_i?

$X_i = 1$ iff i^{th} student gets own HW back

LOE: $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: $\mathbb{E}[X_i] = \frac{1}{n}$

Therefore, $\mathbb{E}[X] = n \cdot \frac{1}{n} = 1$

Pairs with the same birthday

$m = 6$

- In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays; diff people independent)?

March 5: A, D, E — AD, AE, DE

Apr 2: B, F — BF

May 8: C

X: # pairs of people that have same bday

$$\boxed{E(X) \underset{\text{LOE}}{=} \sum_{\substack{\text{all}\\\text{unordered}\\\text{pairs}}} E(X_{ij}) = \binom{m}{2}\frac{1}{365}}$$

$$\underset{\text{unordered pair}}{X_{ij}} = \begin{cases} 1 & \text{if person } i \text{ \& person } j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$$

$$\boxed{X = \sum_{\substack{\text{all unordered}\\\text{pairs of people } ij}} X_{ij}}$$

$$E(X_{ij}) = P(i \,\&\, j \text{ have } \ldots)$$

$$\sum^{365} \Pr(\text{both } i \,\&\, j \ldots)$$

same bday

$$= \frac{1}{365}$$

$$\sum_{k=1} \text{(have bday } k\text{)}$$

$$= \sum_{k=1}^{365} P(i \text{ has bday } k)\, P(j \text{ has bday } k)$$

$$= \sum_{k=1}^{365} \frac{1}{365} \cdot \frac{1}{365} = \frac{1}{365}$$

Pairs with the same birthday

- In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?

Decompose: Indicator events involve **pairs** of students (i, j) for $i \neq j$

$X_{ij} = 1$ iff students i and j have the same birthday

LOE: $\binom{m}{2}$ indicator variables X_{ij}

Conquer: $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365} = \frac{m(m-1)}{730}$ pairs

Agenda

- Recap
- Linearity of expectation
- LOTUS
- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Linearity of Expectation – Even stronger

Theorem. For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}[\underbrace{a_1 X_1 + \cdots + a_n X_n}_{g(X_1, \ldots X_n)}] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$

Very important: In general, we do <u>not</u> have $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

ω

$$Z = X \cdot Y$$

$$Z(\omega) = X(\omega) \cdot Y(\omega)$$

$aX+b$

Linearity is special!

In general $\mathbb{E}[g(X)] \neq \boxed{g(\mathbb{E}(X))}$

$E(X) = 0$

E.g., $X = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$

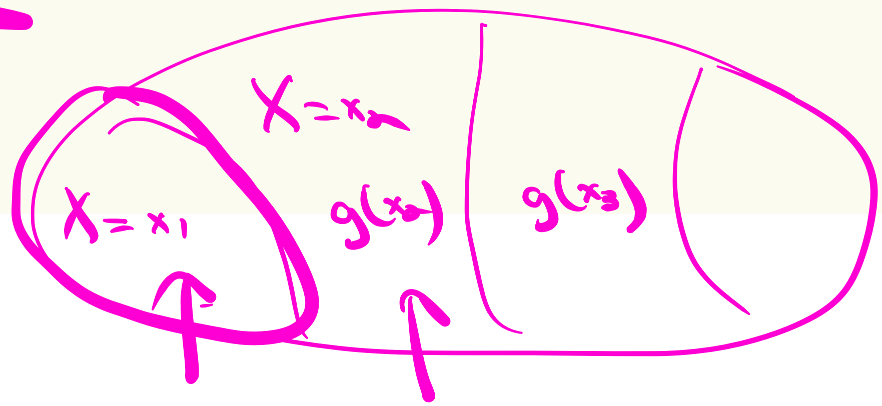
0 otherwise

X^2 is 1 with prob 1

Then: $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

$E(X^2) = 1$

How DO we compute $\mathbb{E}[g(X)]$?



$g(X)$

$$\mathbb{E}(g(X)) = g(x_1)P(X=x_1) + g(x_2)P(X=x_2) + \dots$$

Expected Value of $g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\Rightarrow \quad \mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[g(X)] = \sum_{x \in X(\Omega)} g(x) \cdot P(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Also known as **LOTUS:** "Law of the unconscious statistician

(nothing special going on in the discrete case)

Example: Expectation of $g(X)$

$W = 10X^3$

Suppose we rolled a fair, 6-sided die in a game.
You will win the cube of the number rolled in dollars, times 10.
Let X be the result of the dice roll.
What is your expected winnings?

$g(i)$

$E(W)$

$= \mathbb{E}[10X^3] = \sum_{i=1}^{6} 10 i^3 \, P(X=i)$

$10 \, 0^3 \, P(X=0) + 10 \cdot 1^3 \, P(X=1) + 10 \cdot 2^3 \, P(X=2)$

$+ \ldots \; 10 \, 6^3 \, P(X=6)$