CSE 312
Foundations of Computing II

Lecture 8: More on random variables; expectation
**Last Class:**
- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)

**Today:**
- Recap
- Expectation
- Linearity of Expectation
- Indicator Random Variables
**Definition.** A random variable (RV) for a probability space \((\Omega, P)\) is a function \(X: \Omega \rightarrow \mathbb{R}\).

The set of values that \(X\) can take on is its range/support: \(X(\Omega)\) or \(\Omega_X\).
Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

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<th>$\Pr(\omega)$</th>
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**Definition.** A random variable (RV) for a probability space $(\Omega, P)$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that $X$ can take on is its range/support: $X(\Omega)$ or $\Omega_X$

$$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$$

Random variables partition the sample space.

$$\Sigma_{x \in X(\Omega)} P(X = x) = 1$$
Example: Returning Homeworks

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$\Omega$

$X(\omega) = 0$

- $\{2, 3, 1\}$
- $\{3, 1, 2\}$

$X(\omega) = 1$

- $\{1, 3, 2\}$
- $\{2, 1, 3\}$
- $\{3, 1, 2\}$

$X(\omega) = 3$

- $\{1, 2, 3\}$
Review PMF and CDF

Definitions:

For a RV $X: \Omega \to \mathbb{R}$, the **probability mass function (pmf)** of $X$ specifies, for any real number $x$, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\sum_{x \in \Omega_X} p_X(x) = 1$$

For a RV $X: \Omega \to \mathbb{R}$, the **cumulative distribution function (cdf)** of $X$ specifies, for any real number $x$, the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$
Example – Two fair independent coin flips

\[ X = \text{number of heads} \]
Example: Returning Homeworks

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$$P(X = 0) = \frac{1}{3}$$
$$P(X = 1) = \frac{1}{2}$$
$$P(X = 3) = \frac{1}{6}$$
Example – Number of Heads

We flip $n$ coins, independently, each heads with probability $p$

$\Omega = \{HH \ldots HH, HH \ldots HT, HH \ldots TH, \ldots, TT \ldots TT\}$

$X = \# \text{ of heads}$

$$p_X(k) = P(X = k) =$$
Example – Number of Heads

We flip $n$ coins, independently, each heads with probability $p$

$$\Omega = \{\text{HH \ldots HH, HH \ldots HT, HH \ldots TH, \ldots , TT \ldots TT}\}$$

$X = \# \text{ of heads}$

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k}$$

# of sequences with $k$ heads  

Prob of sequence w/ $k$ heads
Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation
Expectation (Idea)

**Example.** Toss a coin 20 times independently with probability $\frac{1}{4}$ of coming up heads on each toss.

$X = \text{number of heads}$

How many heads do you *expect* to see?

What if you toss it independently $n$ times and it comes up heads with probability $p$ each time?
Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value or mean of $X$ is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)
**Expectation**

**Example.** Two fair coin flips

\[ \Omega = \{TT, HT, TH, HH\} \]

\[ X = \text{number of heads} \]

What is \( \mathbb{E}[X] \)?

\[
\begin{align*}
\mathbb{E}[X] &= 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2) \\
&= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} \\
&= \frac{1}{2} + \frac{1}{2} = 1
\end{align*}
\]
Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW
- What is $\mathbb{E}[X]$?

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$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$

$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x)$
Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

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$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$

$= 6 \cdot \frac{1}{6} = 1$
Example – Flipping a biased coin until you see heads

- Biased coin, each flip indep:
  \[ P(H) = q > 0 \]
  \[ P(T) = 1 - q \]
- \( Z = \# \) of coin flips until first head

\[ P(Z = i) = \]

\[ \mathbb{E}[Z] = \]
Example – Flipping a biased coin until you see heads

• Biased coin, each flip indep:
  \[ P(H) = q > 0 \]
  \[ P(T) = 1 - q \]

• \( Z = \# \) of coin flips until first head

\[
P(Z = i) = q (1 - q)^{i-1}
\]

\[
\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot P(Z = i) = \sum_{i=1}^{\infty} i \cdot q (1 - q)^{i-1}
\]

Converges, so \( \mathbb{E}[Z] \) is finite

Can calculate this directly but...
Example – Flipping a biased coin until you see heads

• Biased coin, each flip indep:
  \[ P(H) = q > 0 \]
  \[ P(T) = 1 - q \]

• \( Z \) = # of coin flips until first head

Another view: If you get heads first try you get \( Z = 1 \); If you get tails you have used one try and have the same experiment left

\[ \mathbb{E}[Z] = \]
Example – Flipping a biased coin until you see heads

• Biased coin:
  \( P(H) = q > 0 \)
  \( P(T) = 1 - q \)

• \( Z \) = # of coin flips until first head

   \[ Z = \begin{cases} 1 & \text{if heads first try} \\ 1 & \text{if tails first try} \end{cases} \]

   Another view: If you get heads first try you get \( Z = 1 \);
   If you get tails you have used one try and have the same experiment left

\[ \mathbb{E}[Z] = q \cdot 1 + (1 - q)(1 + \mathbb{E}(Z)) \]

Solving gives \( q \cdot \mathbb{E}[Z] = q + (1 - q) = 1 \) \( \implies \mathbb{E}[Z] = 1/q \)
Example – Coin Tosses

We flip $n$ coins, each toss independent, probability $p$ of coming up heads.

$Z$ is the number of heads, what is $\mathbb{E}(Z)$?
Example – Coin Tosses

We flip \( n \) coins, each toss independent; heads with probability \( p \), \( Z \) is the number of heads, what is \( \mathbb{E}[Z] \)?

\[
\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}
\]

\[
= \sum_{k=0}^{n} k \cdot \frac{n!}{k! (n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)! (n-k)!} p^k (1-p)^{n-k}
\]

\[
= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)! (n-k)!} p^{k-1} (1-p)^{n-k}
\]

\[
= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k! (n-1-k)!} p^k (1-p)^{(n-1)-k}
\]

\[
= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np (p + (1-p))^{n-1} = np \cdot 1 = np
\]
Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
• Linearity of Expectation
**Linearity of Expectation**

**Theorem.** For any two random variables $X$ and $Y$
(no conditions whatsoever on the random variables)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \ldots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

**Because:**

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[(X_1 + \cdots + X_{n-1}) + X_n]$$

$$\hspace{1cm} = \mathbb{E}[X_1 + \cdots + X_{n-1}] + \mathbb{E}[X_n] = \cdots$$
Linearity of Expectation – Proof

**Theorem.** For any two random variables $X$ and $Y$ ($X, Y$ do not need to be independent)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$$\mathbb{E}[X + Y] = \sum_\omega P(\omega)(X(\omega) + Y(\omega))$$

$$= \sum_\omega P(\omega)X(\omega) + \sum_\omega P(\omega)Y(\omega)$$

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$
Using LOE to compute complicated expectations

Often boils down to the following three steps:

- **Decompose:** Finding the right way to decompose the random variable into sum of simple random variables
  \[ X = X_1 + \cdots + X_n \]
- **LOE:** Apply linearity of expectation.
  \[ \mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]. \]
- **Conquer:** Compute the expectation of each \( X_i \)

Often, \( X_i \) are **indicator** (0/1) random variables.
Indicator random variables – 0/1 valued

For any event $A$, can define the indicator random variable $X_A$ for $A$

$$X_A = \begin{cases} 
1 & \text{if event } A \text{ occurs} \\
0 & \text{if event } A \text{ does not occur}
\end{cases}$$

$$P(X_A = 1) = P(A)$$

$$P(X_A = 0) = 1 - P(A)$$
Example – Coin Tosses – The brute force method

We flip \( n \) coins, each one heads with probability \( p \),
\( Z \) is the number of heads, what is \( \mathbb{E}[Z] \)?

\[
\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1 - p)^{n-k}
\]

\[
= \sum_{k=0}^{n} k \cdot \frac{n!}{k! (n - k)!} p^k (1 - p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k - 1)! (n - k)!} p^k (1 - p)^{n-k}
\]

\[
= np \sum_{k=1}^{n} \frac{(n - 1)!}{(k - 1)! (n - k)!} p^{k-1} (1 - p)^{n-k}
\]

\[
= np \sum_{k=0}^{n-1} \frac{(n - 1)!}{k! (n - 1 - k)!} p^k (1 - p)^{(n-1)-k}
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\[
= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1 - p)^{(n-1)-k} = np (p + (1 - p))^{n-1} = np \cdot 1 = np
\]

Can we solve it more elegantly, please?
Computing complicated expectations

Often boils down to the following three steps:

- **Decompose:** Finding the right way to decompose the random variable into sum of simple random variables

  \[ X = X_1 + \cdots + X_n \]

- **LOE:** Apply linearity of expectation.

  \[ \mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]. \]

- **Conquer:** Compute the expectation of each \( X_i \)

Often, \( X_i \) are indicator (0/1) random variables.
Example – Coin Tosses

We flip $n$ coins, each toss independent, comes up heads with probability $p$

$Z$ is the number of heads, what is $\mathbb{E}[Z]$?

$$X_i = \begin{cases} 1, & \text{$i^{th}$ coin flip is heads} \\ 0, & \text{$i^{th}$ coin flip is tails.} \end{cases}$$

\[ Z = X_1 + \cdots + X_n \]

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Example – Coin Tosses

We flip $n$ coins, each toss independent, comes up heads with probability $p$

$Z$ is the number of heads, what is $\mathbb{E}[Z]$?

$X_i = \begin{cases} 1, & \text{$i^{th}$ coin flip is heads} \\ 0, & \text{$i^{th}$ coin flip is tails.} \end{cases}$

Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n \cdot p$$

$P(X_i = 1) = p$

$P(X_i = 0) = 1 - p$

$$\mathbb{E}[X_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$
Kandinsky

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Example: Returning Homeworks

- Class with $n$ students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

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**Decompose:** Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

**LOE:** Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

**Conquer:** Compute the expectation of each $X_i$ and sum!
Example: Returning Homeworks

• Class with $n$ students, randomly hand back homeworks. All permutations equally likely.
• Let $X$ be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

**Decompose:**

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**LOE:**

**Conquer:**
Example: Returning Homeworks

• Class with \( n \) students, randomly hand back homeworks. All permutations equally likely.
• Let \( X \) be the number of students who get their own HW

What is \( \mathbb{E}[X] \)? Use linearity of expectation!

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**Decompose:** What is \( X_i \)?

\( X_i = 1 \) iff \( i^{th} \) student gets own HW back

**LOE:** \( \mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \)

**Conquer:** \( \mathbb{E}[X_i] = \frac{1}{n} \)

Therefore, \( \mathbb{E}[X] = n \cdot \frac{1}{n} = 1 \)
Pairs with the same birthday

- In a class of $m$ students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
Pairs with the same birthday

- In a class of $m$ students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?

**Decompose:** Indicator events involve pairs of students $(i, j)$ for $i \neq j$

$$X_{ij} = 1 \text{ iff students } i \text{ and } j \text{ have the same birthday}$$

**LOE:** $\binom{m}{2}$ indicator variables $X_{ij}$

**Conquer:** $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365} = \frac{m(m-1)}{730}$ pairs
Linearity of Expectation – Even stronger

**Theorem.** For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$