CSE 312

Foundations of Computing II

Lecture 2: Permutations, combinations, the Binomial Theorem and more.

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

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Slide Style Credit: Based on Stefano Tessaro’s slides for 312 19au
incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Grading, syllabus and administrivia

- Questions?

Agenda

- Recap & Examples ⬅
- Binomial Theorem
- Multinomial Coefficients
- Inclusion-Exclusion
- Combinatorial Proofs

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Quick Summary

- Sum Rule

  If you can choose from
  - Either one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- Product Rule

  In a sequential process, if there are
  - n_1 choices for the first step,
  - n_2 choices for the second step (given the first choice), …, and
  - n_k choices for the k^{th} step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_k$

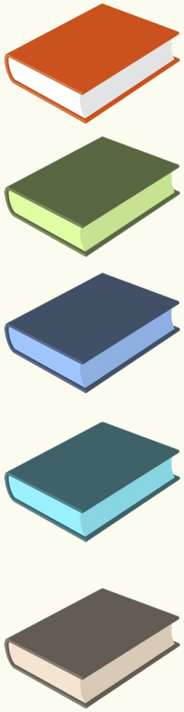
- Complementary Counting

Quick Summary

- K-sequences: How many length k sequences over alphabet of size n? repetition allowed.
  - Product rule → n^K
- K-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation → $\frac{n!}{(n-k)!}$
- K-combinations: How many size k subsets of a set of n distinct elements (without repetition and without order)?
  - Combination → $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Product rule – Another example

5 books



> ***"How many ways are there to distribute 5 books among Alice, Bob, and Charlie?"***

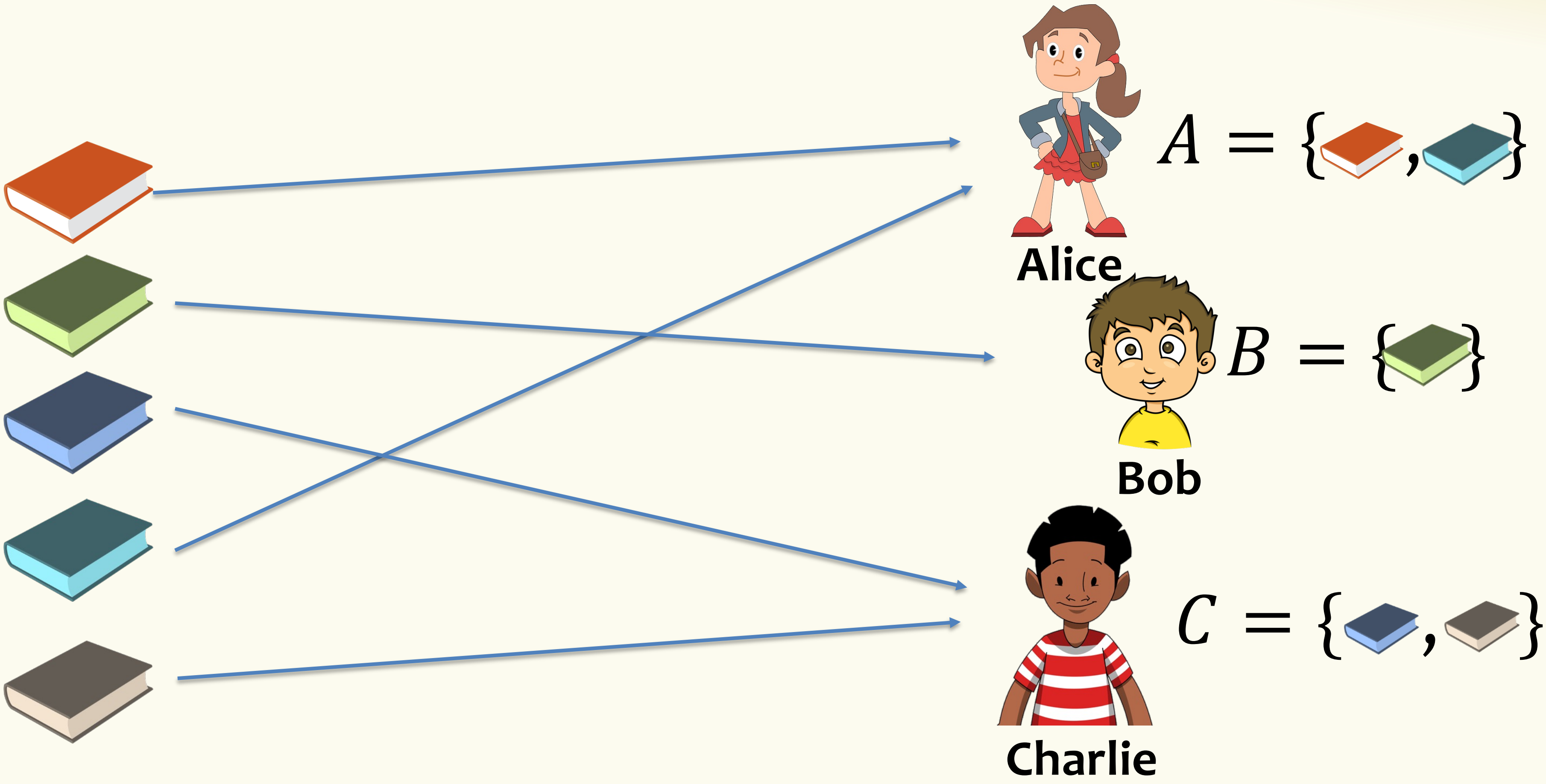
Every book to one person, everyone gets ≥ 0 books.

Alice

Bob

Charlie

Example Book Assignment



Book assignment – Modeling

Correct?

Poll:

A. right

B. Overcount

C. Undercount

D. No idea

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$\boxed{2^5 = 32 \text{ options}}$

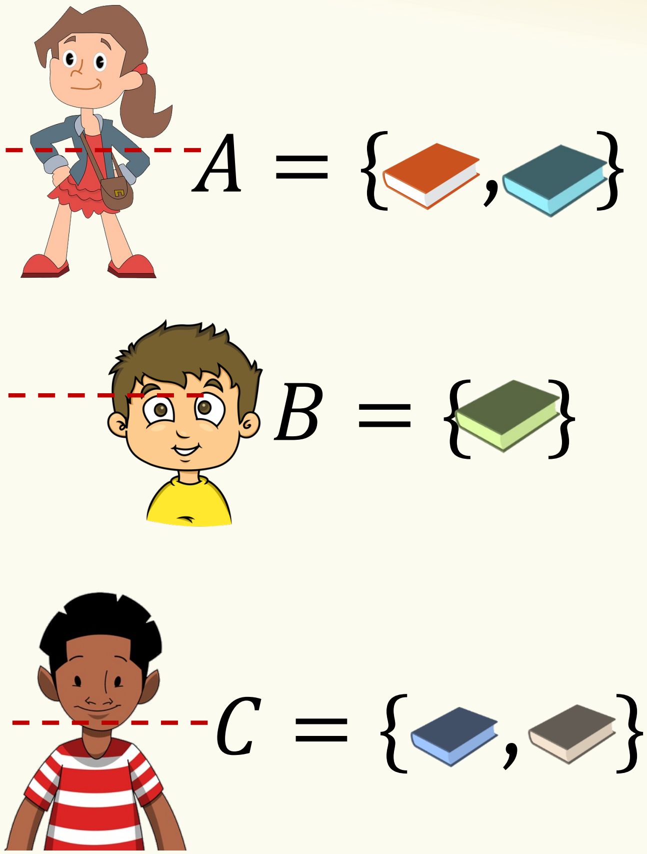
$\times$

$\boxed{2^5 = 32 \text{ options}}$

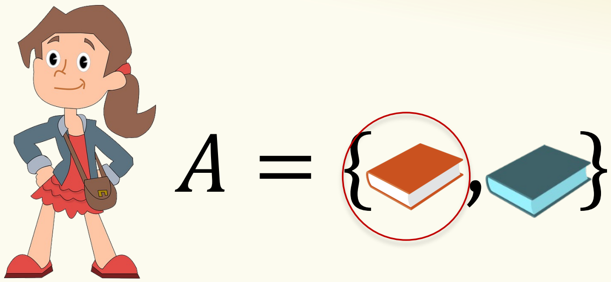
$\times$

$\boxed{2^5 = 32 \text{ options}}$

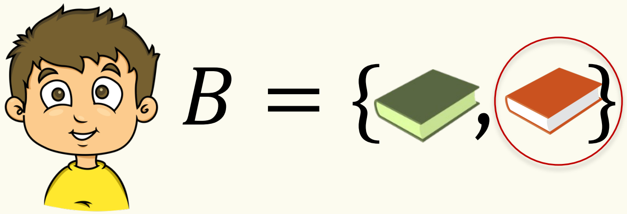
$= \boxed{32^3 \text{ assignment}}$



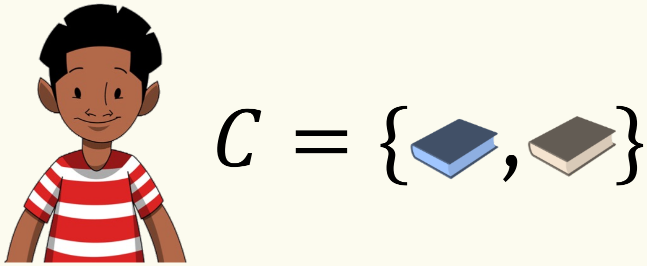
Problem – Overcounting



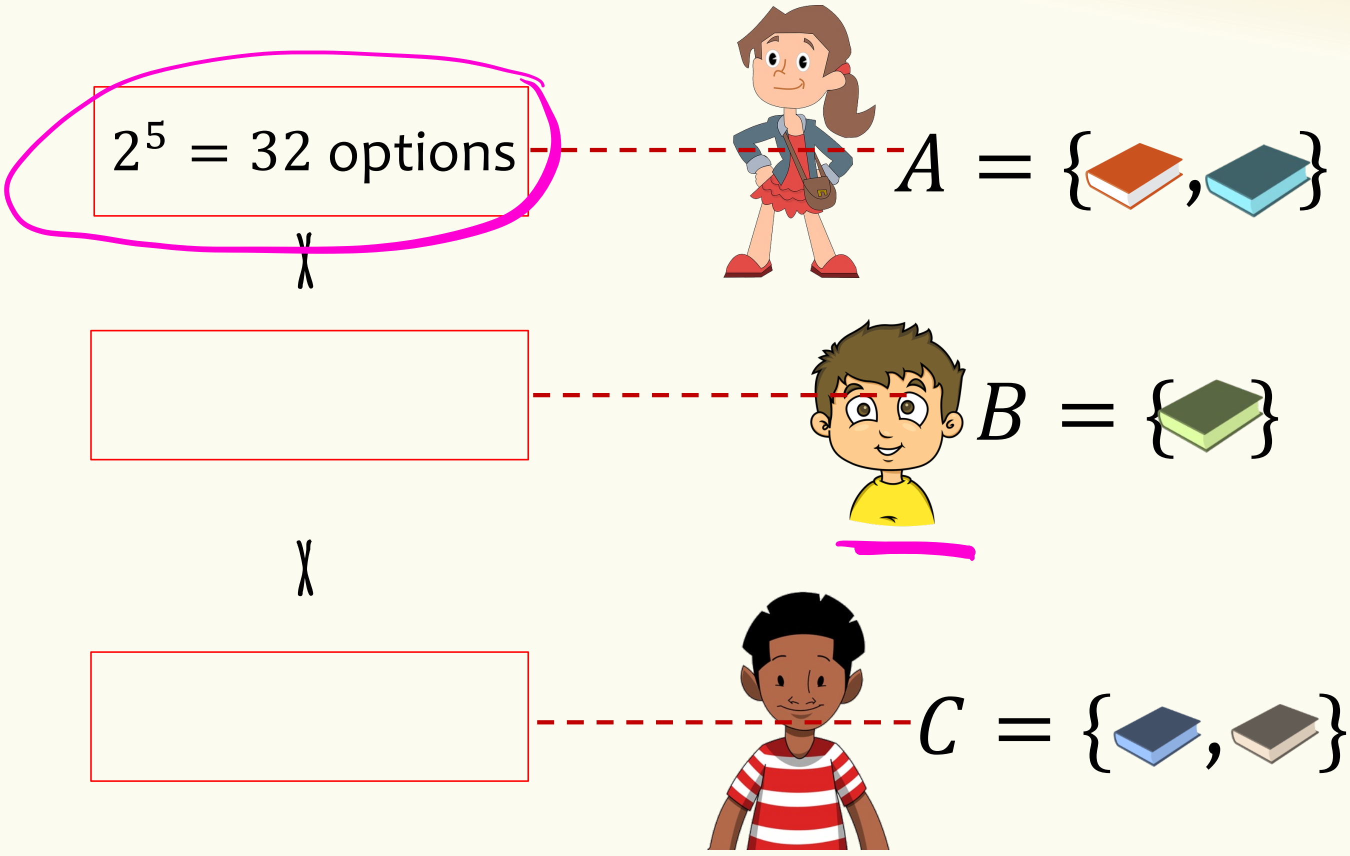
Problem: We are counting some invalid assignments!!!
→ **overcounting**!



What went wrong in the sequential process?
After assigning set A to Alice, set B is no longer a valid option for Bob



Book assignment – Second try



$2^5 = 32$ options

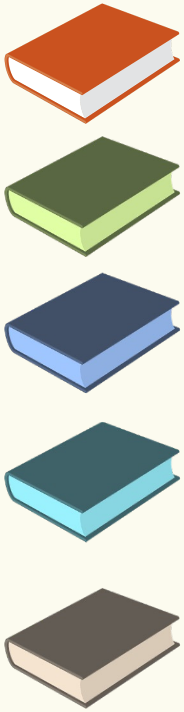
$A = \{\ ,\ \}$

$B = \{\ \}$

$C = \{\ ,\ \}$

Product rule – A better way

5 books



“How many ways are there to distribute 5 books among Alice, Bob, and Charlie?”

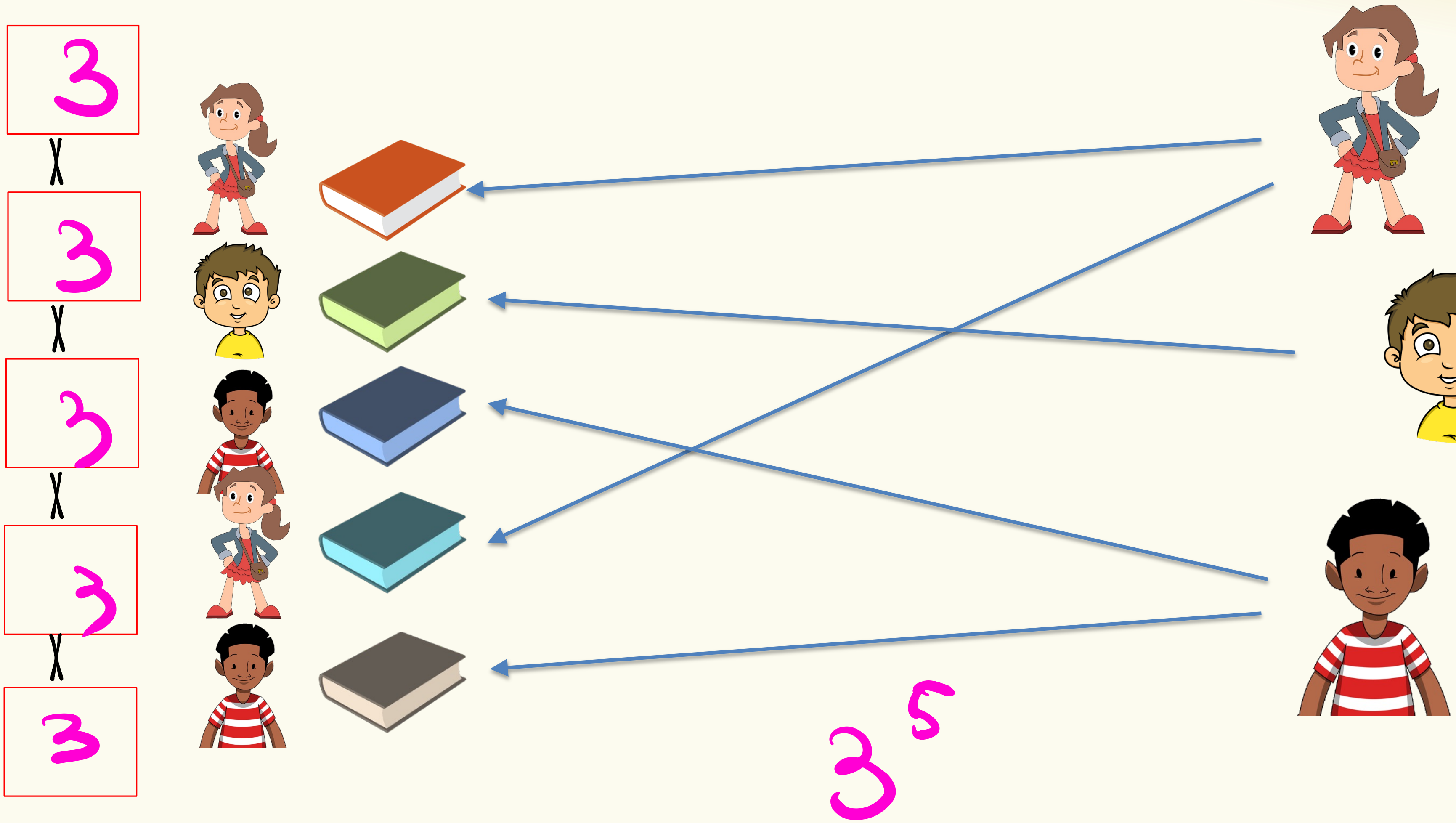
Every book to one person, everyone gets ≥ 0 books.

Alice

Bob

Charlie

Book assignments – Choices tell you who gets each book



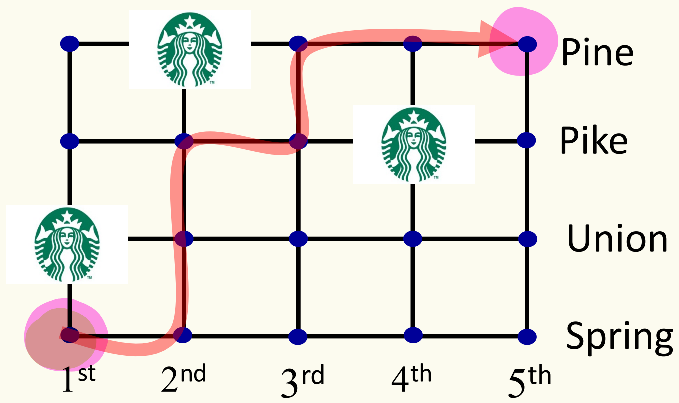
$3 \times 3 \times 3 \times 3 \times 3$

3^5

Lesson: Representation of what we are counting is very important!

Think about the various possible ways you could make a sequence of choices that leads to an outcome in the set of outcomes you are trying to count.

Example – Counting Paths

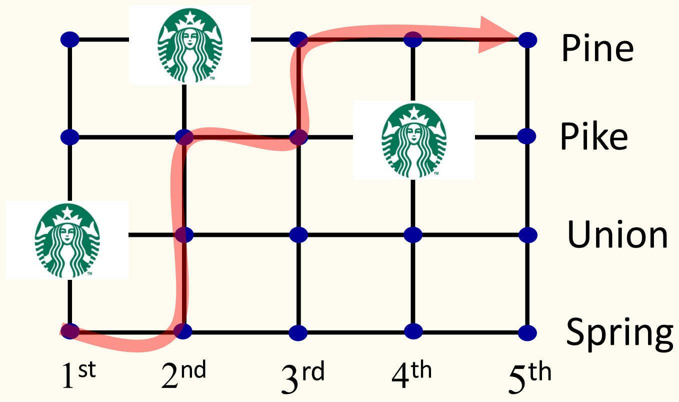


"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?

↑ →

→ ↑ ↑ → ↑ → →

Example – Counting Paths -2



0

“How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?

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Poll:

→ A. 2^7

B. $\frac{7!}{4!}$

[✓ C.] $\binom{7}{4} = \frac{7!}{4!3!}$

[✓ D.] $\binom{7}{3} = \frac{7!}{3!4!}$

2 → 3 → 6 → 7 →

0

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

Why??

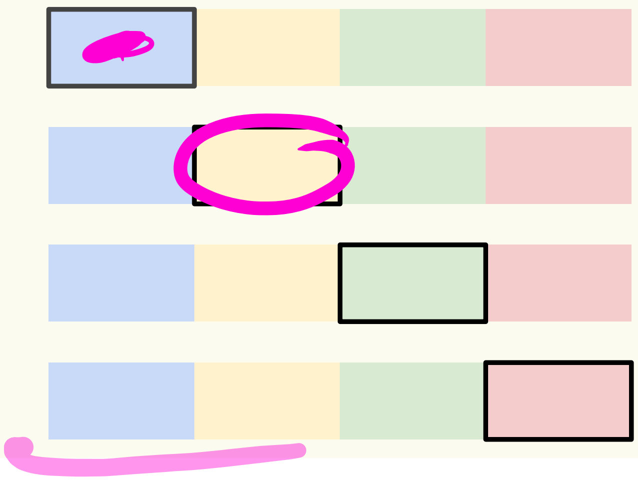
This is called an Algebraic proof, i.e., Prove by checking algebra

Symmetry in Binomial Coefficients – A different proof

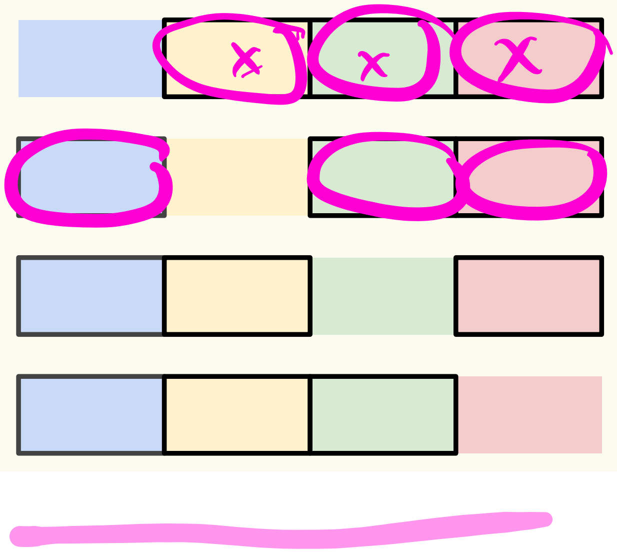
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$



n=4 k=1

Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

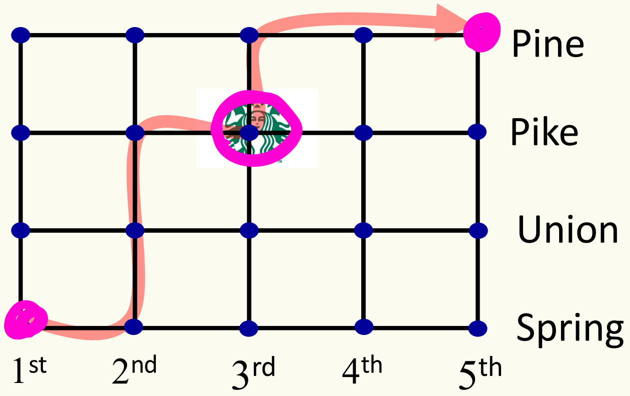
1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

This is called a combinatorial argument/proof

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = N$
- Show how to count $|S|$ another way => $|S| = m$

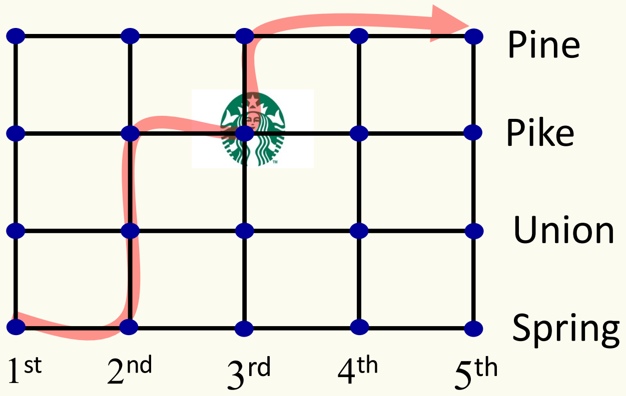
More examples of combinatorial proofs coming soon!

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ *and* → *but stopping at Starbucks on 3rd and Pike?"*

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → but stopping at Starbucks on 3rd and Pike?"

Poll:

- *A.* $\binom{7}{3}$
- *B.* $\binom{7}{3}\binom{7}{1}$
- *C.* $\binom{4}{2}\binom{3}{1}$
- *D.* $\binom{4}{2}\binom{3}{2}$

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- Recap & Examples
- Binomial Theorem ◀
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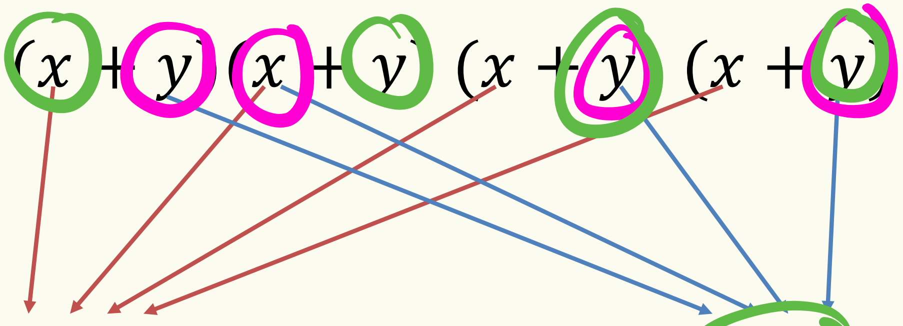
Binomial Theorem: Idea

$(x+y)^n$

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \dots$$

xy^3

$$\_\,x^4 + \_\,x^3y + \_\,x^2y^3 + \_\,xy^3 + \_\,y^4$$

$\uparrow$

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

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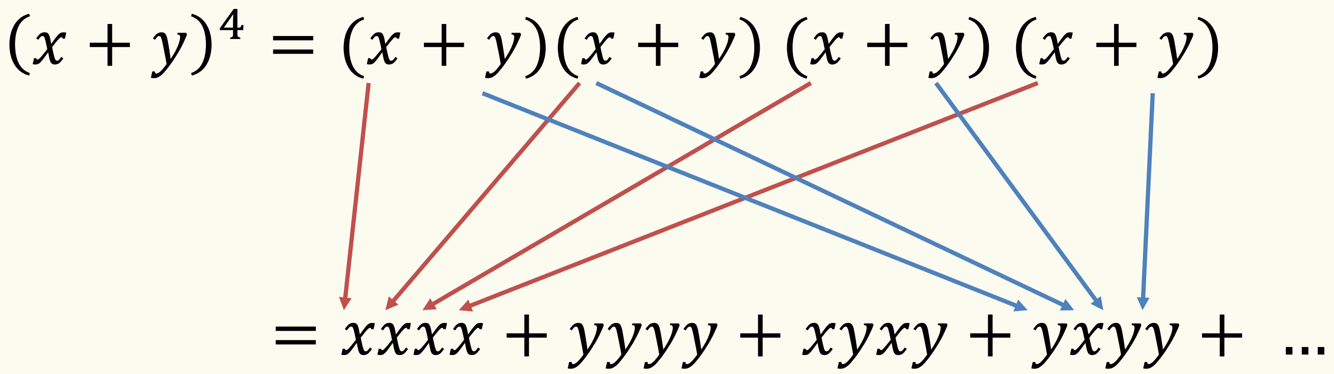

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$

$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$?

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\Uparrow$$

Binomial Theorem: Idea

$$(x+y)^n = (x+y)(x+y)(x+y)\cdots(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of the n variables we multiple to be an x (the rest will be y).

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{\boxed{k=0}}^{n} \binom{n}{k} \underline{x^k y^{n-k}}$$

$x^0 y^n$ $x^1 y^{n-1}$

$k=0$ $k=1$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$(1+1)^n$

$= 2^n$

$y^k x^{n-k}$

$x = 1$

$y = 1$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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Example – Word Permutations

How many ways to re-arrange the letters in the word "MATH"?

Poll:

A. $\binom{26}{4}$

B. 4^4

[✓ C. $4!$]

D. *I don't know*

MATH

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Example – Word Permutations

How many ways to re-arrange the letters in the word “MUUMUU”?

6!

______ ______ ______ ______ ______ ______

Example – Word Permutations

How many ways to re-arrange the letters in the word “MUUMUU”?

_ M _ _ M _

Choose where the 2 M’s go, and then the U’s are set **OR**
Choose where the 4 U’s go, and then the M’s are set

Either way, we get $\underline{\binom{6}{2} \cdot \binom{4}{4}} = \binom{6}{4} \cdot \binom{2}{2} = \boxed{\frac{6!}{2!4!}}$

Another way to think about it

How many ways to re-arrange the letters in the word "MUUMUU"?

Arrange the 6 letters as if they were distinct.

$M\,U_1U\,M_2U_3U_4$

Then divide by 4! to account for duplicate M's and divide by 2! to account for duplicate U's.

Yields $\boxed{\frac{6!}{2!4!}}$

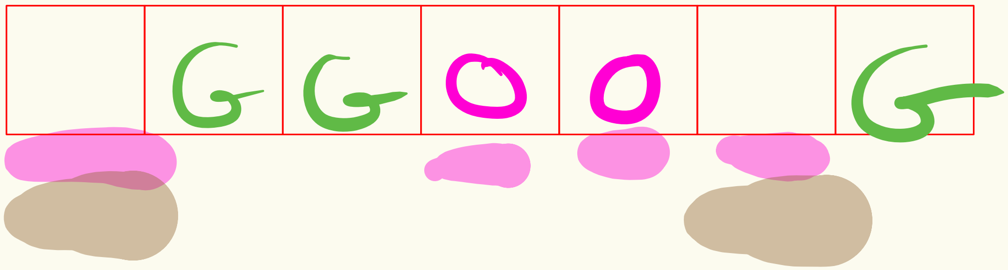
6!

MU UMUU

Another example – Word Permutations

How many ways to re-arrange the letters in the word "GODOGGY"?

3 G's
2 O's
1 D
1 Y



Poll:

- A. 7!
- B. $\frac{7!}{3!}$
- C. $\frac{7!}{3!2!1!1!}$
- D. $\binom{7}{3} \cdot \binom{4}{2} \cdot 2!$

G's

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Multinomial coefficients

4 3 G's 2U's

If we have k types of objects, with n_1 of the first type, n_2 of the second type, … , n_k of the kth type, where $n = n_1 + n_2 + \cdots + n_k$ then the number of arrangements of the n objects is

7 ✓

$$\underline{\binom{n}{n_1, n_2, \dots, n_k}} = \frac{\boxed{n!}}{\boxed{n_1}!\, n_2! \cdots n_k\,!}$$

Note that objects of the same type are indistinguishable.

Example – Word Permutations

How many ways to re-arrange the letters in the word “GODOGGY”?

$n= 7$ (length of sequence) $K = 4$ types = {G, O, D, Y}

$n_1 = \underline{3}, n_2 = \underline{2}, n_3 = 1, n_4 = 1$

$$\binom{6}{4,2,1,1} = \frac{6!}{2!4!1!1!}$$

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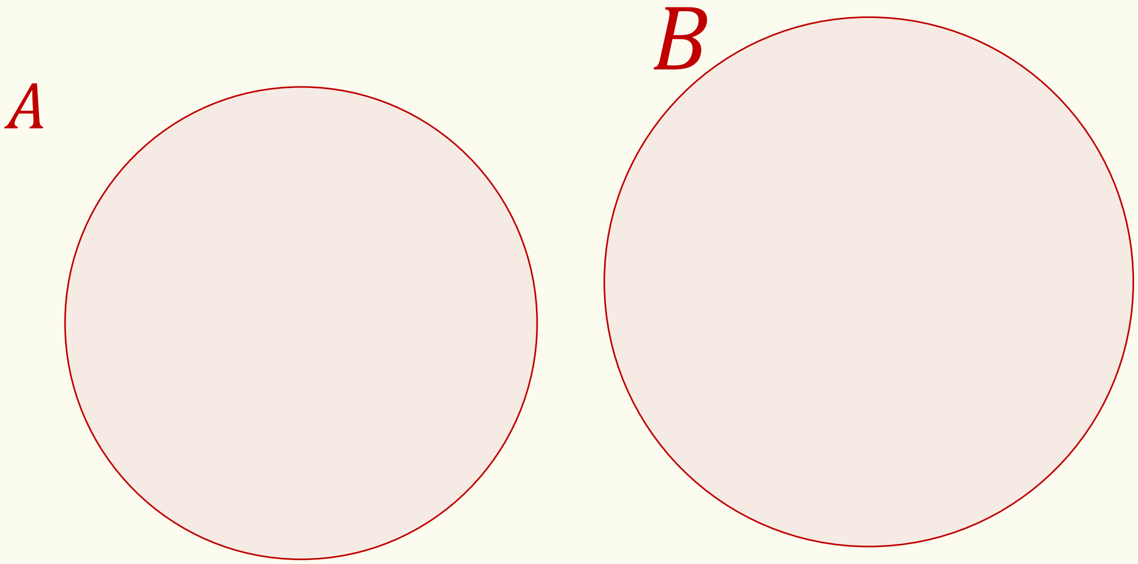
Recap Disjoint Sets

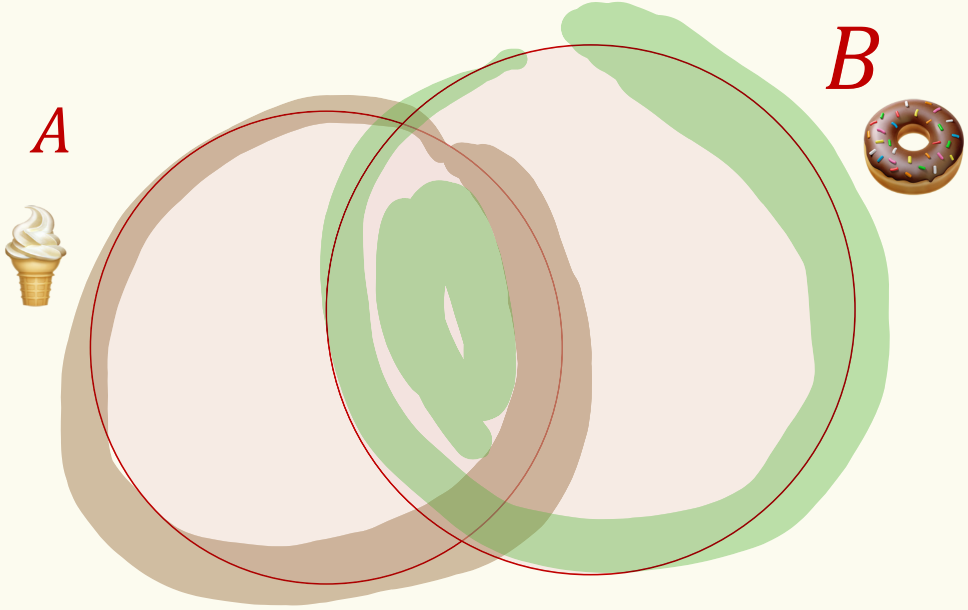
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?



$|A| = 43$

$|B| = 20$

$|A \cap B\ | = 7$

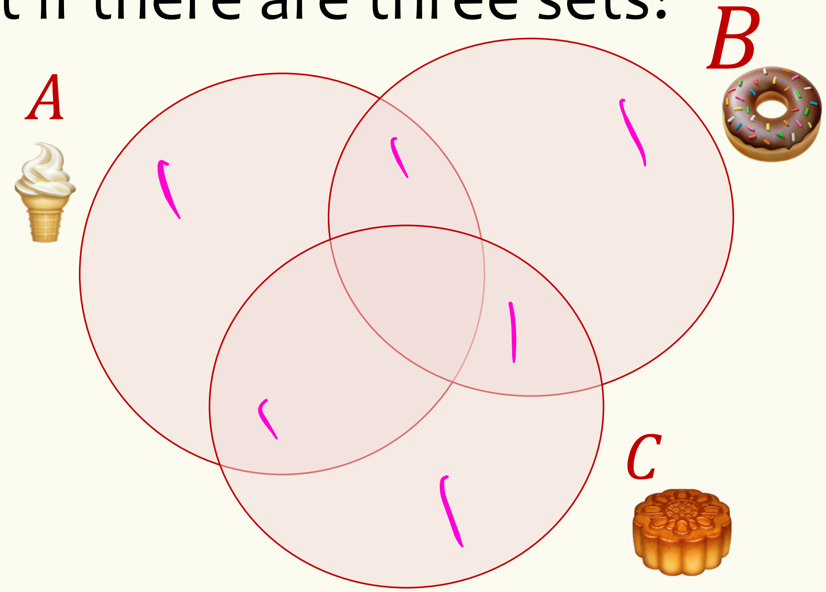
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B| = 7$
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

Fact.

$$\begin{aligned}|A \cup B \cup C| = {} & \underline{|A|} + \underline{|B|} + |C| \\ & - \underline{|A \cap B|} - \underline{|A \cap C|} - \underline{|B \cap C|} \\ & + |A \cap B \cap C|\end{aligned}$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \dots, A_n$ are sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= singles\ - doubles + triples\ - quads +\ \dots \\ &= (\underline{|A_1| + \cdots + |A_n|})\ - (|\underline{A_1 \cap A_2}| +\ \dots + |A_{n-1} \cap A_n|) +\ \dots \end{aligned}$$

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Combinatorial proof: Show that $M = N$

$$\underset{\uparrow}{\binom{n}{k}} = \underset{\uparrow}{\binom{n}{n-k}}$$

- Let $\underline{S}$ be a set of objects
- Show how to count $|S|$ one way => $\underline{|S|} = \boxed{M}$
- Show how to count $|S|$ another way => $\underline{|S|} = \boxed{N}$
- Conclude that $\underline{M = N}$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

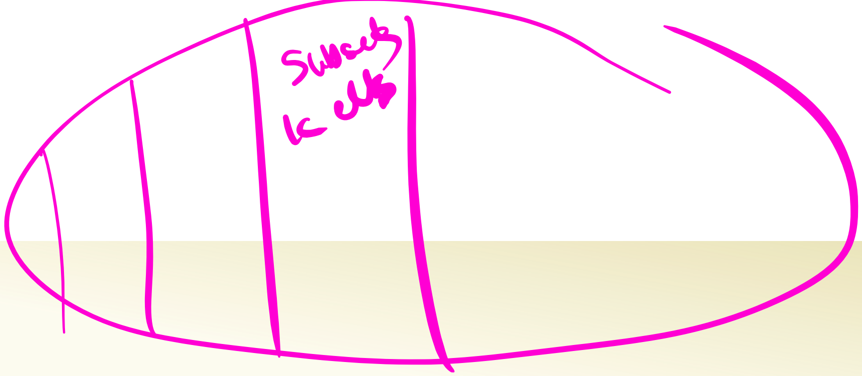
⇒ **Fact.** $\binom{n}{k} = \binom{n}{n-k}$ Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ Pascal's Identity ⇙

⇒ **Fact.** $\underbrace{\sum_{k=0}^{n} \binom{n}{k}} = \boxed{2^n}$ Follows from Binomial theorem

⇑

$\binom{n}{k}$ # subsets w/ k elts



Pascal’s Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal’s identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= 20\ \textit{years later} \ldots$$

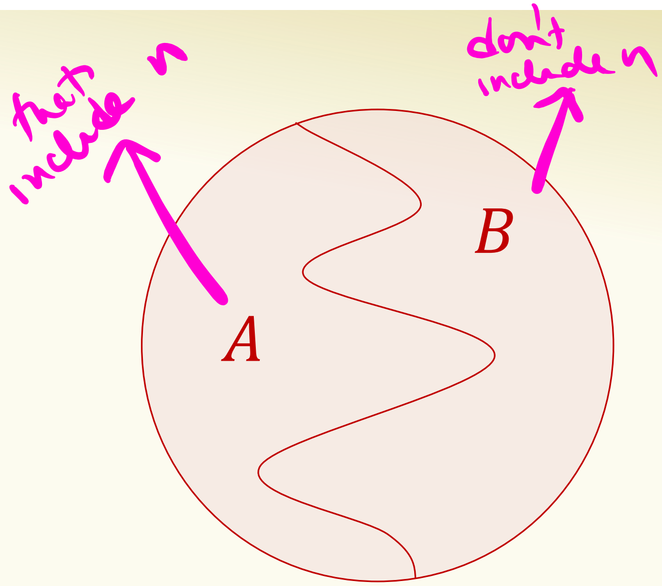
$$= \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}$$

Hard work and not intuitive

Let’s see a combinatorial argument

Example – Binomial Identity



Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\begin{array}{ccccc} \binom{n}{k} & = & \binom{n-1}{k-1} & + & \binom{n-1}{k} \\ \uparrow & & \uparrow & & \uparrow \\ |S| & = & |A| & + & |B| \end{array}$$

$S = A \cup B$, disjoint

$\underline{S}$: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $\underline{|S| = \binom{n}{k}}$

A: the set of size k subsets of $[n]$ including n

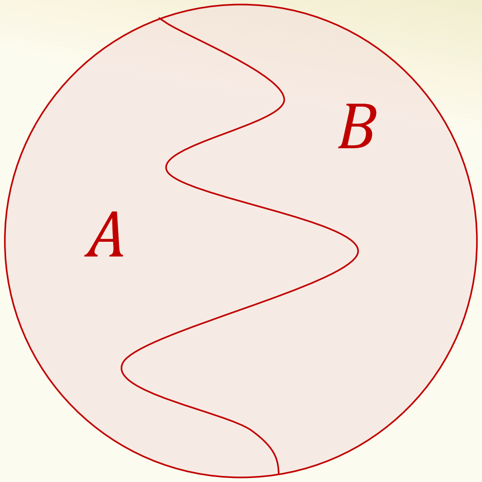
B: the set of size k subsets of $[n]$ NOT including n

Sum rule:
$|A \cup B| = |A| + |B|$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4$, $S = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

k=2.

A: the set of size k subsets of $[n]$ including n

$A = \{\{1,4\},\{2,4\},\{3,4\}\}$. $n = 4$

B: the set of size k subsets of $[n]$ NOT including n

$B = \{\{1,2\},\{1,3\},\{2,3\}\}$

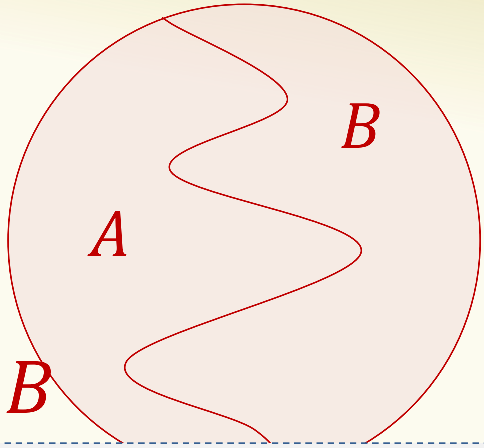
1 2 3 ~~4~~

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S|$ $|A|$ $|B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{\boxed{1, 2, \cdots n-1}, n\}$

A: the set of size k subsets of $[n]$ including n

n is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

B: the set of size k subsets of $[n]$ NOT including n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

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