CSE 312

# Foundations of Computing II

## Lecture 16: Midterm review

**Anonymous questions: www.slido.com/4694375**

## Midterm Monday at 9:30am

- Sections AA, AB, AC, AD ➔ Kane 110
- Sections AE, AF, AG, AH ➔ ARC 147
- Can bring one page cheat sheet – must be physical paper.

- **Bring a laptop with you.**
- **Make sure your laptop is fully charged!!**
- During exam, you can have two windows open
  - One open to canvas
  - One open to wolframalpha
- Everything else must be put away.
- Can submit one time only!

Probability & Statistics with Applications to Computing
Key Definitions and Theorems

# 1 Combinatorial Theory

## 1.1 So You Think You Can Count?

**The Sum Rule:** If an experiment can either end up being one of $N$ outcomes, or one of $M$ outcomes (where there is no overlap), then the total number of possible outcomes is: $N + M$.

**The Product Rule:** If an experiment has $N_1$ outcomes for the first stage, $N_2$ outcomes for the second stage, $\ldots$, and $N_m$ outcomes for the $m^{\text{th}}$ stage, then the total number of outcomes of the experiment is $N_1 \times N_2 \cdot \cdots \cdot N_m = \prod_{i=1}^{m} N_i$.

**Permutation:** The number of orderings of $N$ **distinct** objects is $N! = N \cdot (N-1) \cdot (N-2) \cdot \ldots 3 \cdot 2 \cdot 1$.

**Complementary Counting:** Let $\mathcal{U}$ be a (finite) universal set, and $S$ a subset of interest. Then, $\mid S \mid = \mid \mathcal{U} \mid - \mid \mathcal{U} \setminus S \mid$.

## 1.2 More Counting

**$k$-Permutations:** If we want to *pick* (**order matters**) only $k$ out of $n$ distinct objects, the number of ways to do so is:

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \tfrac{n!}{(n-k)!}$$

**$k$-Combinations/Binomial Coefficients:** If we want to *choose* (**order doesn't matter**) only $k$ out of $n$ distinct objects, the number of ways to do so is:

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

**Multinomial Coefficients:** If we have $k$ distinct types of objects ($n$ total), with $n_1$ of the first type, $n_2$ of the second, ..., and $n_k$ of the $k$-th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

k-1 dw

**Stars and Bars/Divider Method:** The number of ways to distribute $n$ indistinguishable balls into $k$ distinguishable bins is

$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

## 1.3 No More Counting Please

**Binomial Theorem:** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

**Principle of Inclusion-Exclusion (PIE):**
2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$k$ events: singles - doubles + triples - quads + ...

**Pigeonhole Principle:** If there are $n$ pigeons we want to put into $k$ holes (where $n > k$), then at least one pigeonhole must contain at least 2 (or to be precise, $\lceil n/k \rceil$) pigeons.

**Combinatorial Proofs:** To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

# 2 Discrete Probability

## 2.1 Discrete Probability

**Key Probability Definitions:** The **sample space** is the set $\Omega$ of all possible outcomes of an experiment. An **event** is any subset $E \subseteq \Omega$. Events $E$ and $F$ are **mutually exclusive** if $E \cap F = \emptyset$.

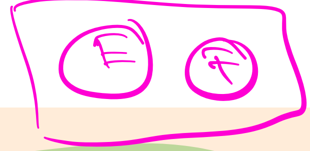
**Axioms of Probability & Consequences:**

1. **(Axiom: Nonnegativity)** For any event $E$, $\mathbb{P}(E) \geq 0$.

counting $\underline{\text{set of objects}}$

choice 1    choice 2    choice k

given outcome
can you uniquely reconstruct what choice was made at each step?




2. **(Axiom: Normalization)** $\mathbb{P}(\Omega) = 1$.
3. **(Axiom: Countable Additivity)** If $E$ and $F$ are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$.

1. **(Corollary: Complementation)** $\mathbb{P}\left(E^C\right) = 1 - \mathbb{P}(E)$
2. **(Corollary: Monotonicity)** If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$
3. **(Corollary: Inclusion-Exclusion)** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

**Equally Likely Outcomes:** If $\Omega$ is a sample space such that each of the unique outcome elements in $\Omega$ are equally likely, then for any event $E \subseteq \Omega$: $\mathbb{P}(E) = |E|/|\Omega|$.

$\forall \omega \in \Omega \quad P(\omega) = \frac{1}{|\Omega|}$

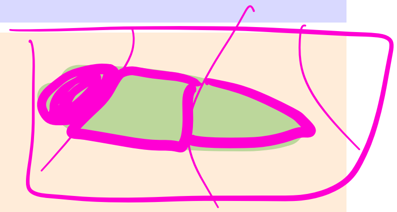
## 2.2 Conditional Probability

**Conditional Probability:** $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

$P(B) > 0$ $\quad P(A \cap B) = \boxed{P(A|B)P(B)}$

**Bayes Theorem:** $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(B \mid A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

**Partition:** Non-empty events $E_1, \ldots, E_n$ **partition** the sample space $\Omega$ if they are both:



- **(Exhaustive)** $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$ (they cover the entire sample space).
- **(Pairwise Mutually Exclusive)** For all $i \neq j$, $E_i \cap E_j = \emptyset$ ( none of them overlap)

Note that for any event $E$, $E$ and $E^C$ always form a partition of $\Omega$.

**Law of Total Probability (LTP):** If events $E_1, \ldots, E_n$ partition $\Omega$, then for any event $F$:

$$\mathbb{P}(F) = \sum_{i=1}^{n} \mathbb{P}(F \cap E_n) = \sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\,\mathbb{P}(E_i)$$

**Bayes Theorem with LTP:** Let events $E_1, \ldots, E_n$ partition the sample space $\Omega$, and let $F$ be another event. Then:

$$\mathbb{P}(E_1 \mid F) = \frac{\mathbb{P}(F \mid E_1)\,\mathbb{P}(E_1)}{\sum_{i=1}^{n} \mathbb{P}(F \mid E_i)\,\mathbb{P}(E_i)}$$

## 2.3 Independence

**Chain Rule:** Let $A_1, \ldots, A_n$ be events with nonzero probabilities. Then:

does not req independence.

$$\mathbb{P}(A_1, \ldots, A_n) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 A_2) \cdots \mathbb{P}(A_n \mid A_1, \ldots, A_{n-1})$$

**Independence:** $A$ and $B$ are **independent** if any of the following equivalent statements hold:

1. $\mathbb{P}(A \mid B) = \mathbb{P}(A)$
2. $\mathbb{P}(B \mid A) = \mathbb{P}(B)$
3. $\mathbb{P}(\underbrace{A, B}_{A \cap B}) = \mathbb{P}(A)\,\mathbb{P}(B)$

**Mutual Independence:** We say $n$ events $A_1, A_2, \ldots, A_n$ are **(mutually) independent** if, for *any* subset $I \subseteq [n] = \{1, 2, \ldots, n\}$, we have

$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i)$$

This equation is actually representing $2^n$ equations since there are $2^n$ subsets of $[n]$.

**Conditional Independence:** $A$ and $B$ are **conditionally independent given an event** $C$ if any of the following equivalent statements hold:

1. $\mathbb{P}(A \mid B, C) = \mathbb{P}(A \mid C)$

2. $\mathbb{P}(B \mid A, C) = \mathbb{P}(B \mid C)$
3. $\mathbb{P}(A, B \mid C) = \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$


# 3 Discrete Random Variables

## 3.1 Discrete Random Variables Basics

**Random Variable (RV):** A random variable (RV) $X$ is a numeric function of the outcome $X : \Omega \to \mathbb{R}$. The set of possible values $X$ can take on is its **range/support**, denoted $\Omega_X$.
If $\Omega_X$ is finite or countable infinite (typically integers or a subset), $X$ is a **discrete RV**. Else if $\Omega_X$ is uncountably large (the size of real numbers), $X$ is a **continuous RV**.

**Probability Mass Function (PMF):** For a discrete RV $X$, assigns probabilities to values in its range. That is $p_X : \Omega_X \to [0, 1]$ where: $p_X(k) = \mathbb{P}(X = k)$.

**Expectation:** The **expectation** of a discrete RV $X$ is: $\mathbb{E}[X] = \sum_{k \in \Omega_X} k \cdot p_X(k)$.

## 3.2 More on Expectation

$E\left(\sum_{i=1}^{n} c_i X_i + c\right)$

**Linearity of Expectation (LoE):** For any random variables $X, Y$ (possibly dependent):

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

$= c_i \sum_{i=1}^{n} E(X_i) + c$

**Law of the Unconscious Statistician (LOTUS):** For a discrete RV $X$ and function $g$, $\mathbb{E}[\underbrace{g(X)}_{Y}] = \sum_{b \in \Omega_X} g(b) \cdot p_X(b)$.

## 3.3 Variance

**Linearity of Expectation with Indicators:** If asked only about the expectation of a RV $X$ which is some sort of "count" (and not its PMF), then you may be able to write $X$ as the sum of possibly dependent **indicator** RVs $X_1, \ldots, X_n$, and apply LoE, where for an indicator RV $X_i$, $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1)$.

**Variance:** $\mathsf{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

**Standard Deviation (SD):** $\sigma_X = \sqrt{\mathsf{Var}(X)}$.

linearity of variance does NOT hold in general

**Property of Variance:** $\mathsf{Var}(aX + b) = a^2\,\mathsf{Var}(X)$.

## 3.4 Zoo of Discrete Random Variables Part I

**Independence:** Random variables $X$ and $Y$ are **independent**, denoted $X \perp Y$, if for *all* $x \in \Omega_X$ and all $y \in \Omega_Y$: $\mathbb{P}(X = x \cap Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$.

$P(X=x \mid Y=y) = P(X=x)$

**Independent and Identically Distributed (iid):** We say $X_1, \ldots, X_n$ are said to be **independent and identically distributed (iid)** if all the $X_i$'s are independent of each other, and have the same distribution (PMF for discrete RVs, or CDF for continuous RVs).

**Variance Adds for Independent RVs:** If $X \perp Y$, then $\mathsf{Var}(X + Y) = \mathsf{Var}(X) + \mathsf{Var}(Y)$.

$E(XY) = E(X)E(Y)$

**Bernoulli Process:** A **Bernoulli process** with parameter $p$ is a sequence of independent coin flips $X_1, X_2, X_3, \ldots$ where $\mathbb{P}(\text{head}) = p$. If flip $i$ is heads, then we encode $X_i = 1$; otherwise, $X_i = 0$.

**Bernoulli/Indicator Random Variable:** $X \sim \text{Bernoulli}(p)$ (Ber$(p)$ for short) iff $X$ has PMF:

$$p_X(k) = \begin{cases} p, & k = 1 \\ 1 - p, & k = 0 \end{cases}$$

$\mathbb{E}[X] = p$ and $\mathsf{Var}(X) = p(1 - p)$. An example of a Bernoulli/indicator RV is one flip of a coin with $\mathbb{P}(\text{head}) = p$. By a clever trick, we can write

$$p_X(k) = p^k (1 - p)^{1-k}, \quad k = 0, 1$$

**Binomial Random Variable:** $X \sim \text{Binomial}(n, p)$ ($\mathsf{Bin}(n, p)$ for short) iff $X$ has PMF

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k \in \Omega_X = \{0, 1, \ldots, n\}$$

$\mathbb{E}[X] = np$ and $\mathsf{Var}(X) = np(1 - p)$. $X$ is the sum of $n$ iid Ber$(p)$ random variables. An example of a Binomial RV is the

$0$, with $np = \lambda$, then $\mathsf{Bin}(n,p) \to \mathsf{Poi}(\lambda)$. If $X_1, \ldots, X_n$ are independent Binomial RV's, where $X_i \sim \text{Bin}(N_i, p)$, then $X = X_1 + \ldots + X_n \sim \text{Bin}(N_1 + \ldots + N_n, p)$.

### 3.5 Zoo of Discrete Random Variables Part II

**Uniform Random Variable (Discrete):** $X \sim \text{Uniform}(a,b)$ ($\text{Unif}(a,b)$ for short), for integers $a \le b$, iff $X$ has PMF:

$$p_X(k) = \frac{1}{b-a+1}, \quad k \in \Omega_X = \{a, a+1, \ldots, b\}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $\mathsf{Var}(X) = \frac{(b-a)(b-a+2)}{12}$. This represents each *integer* in $[a,b]$ to be equally likely. For example, a single roll of a fair die is $\text{Unif}(1,6)$.

**Geometric Random Variable:** $X \sim \text{Geometric}(p)$ ($\text{Geo}(p)$ for short) iff $X$ has PMF:

$$p_X(k) = (1-p)^{k-1} p, \quad k \in \Omega_X = \{1, 2, 3, \ldots\}$$

$\mathbb{E}[X] = \frac{1}{p}$ and $\mathsf{Var}(X) = \frac{1-p}{p^2}$. An example of a Geometric RV is the number of independent coin flips up to and including the first head, where $\mathbb{P}(\text{head}) = p$.

**Negative Binomial Random Variable:** $X \sim \text{NegativeBinomial}(r,p)$ ($\text{NegBin}(r,p)$ for short) iff $X$ has PMF:

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k \in \Omega_X = \{r, r+1, r+2, \ldots\}$$

$\mathbb{E}[X] = \frac{r}{p}$ and $\mathsf{Var}(X) = \frac{r(1-p)}{p^2}$. $X$ is the sum of $r$ iid $\text{Geo}(p)$ random variables. An example of a Negative Binomial RV is the number of independent coin flips up to and including the $r$-th head, where $\mathbb{P}(\text{head}) = p$. If $X_1, \ldots, X_n$ are independent Negative Binomial RV's, where $X_i \sim \text{NegBin}(r_i, p)$, then $X = X_1 + \ldots + X_n \sim \text{NegBin}(r_1 + \ldots + r_n, p)$.

### 3.6 Zoo of Discrete Random Variables Part III

**Poisson Random Variable:** $X \sim \text{Poisson}(\lambda)$ ($\text{Poi}(\lambda)$ for short) iff $X$ has PMF:

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \in \Omega_X = \{0, 1, 2, \ldots\}$$

$\mathbb{E}[X] = \lambda$ and $\mathsf{Var}(X) = \lambda$. An example of a Poisson RV is the number of people born during a particular minute, where $\lambda$ is the average birth rate per minute. If $X_1, \ldots, X_n$ are independent Poisson RV's, where $X_i \sim \text{Poi}(\lambda_i)$, then $X = X_1 + \ldots + X_n \sim \text{Poi}(\lambda_1 + \ldots + \lambda_n)$.

**Hypergeometric Random Variable:** $X \sim \text{HyperGeometric}(N, K, n)$ ($\text{HypGeo}(N, K, n)$ for short) iff $X$ has PMF:

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}, \quad k \in \Omega_X = \{\max\{0, n+K-N\}, \ldots, \min\{K, n\}\}$$

$\mathbb{E}[X] = n\frac{K}{N}$ and $\mathsf{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$. This represents the number of successes drawn, when $n$ items are drawn from a bag with $N$ items ($K$ of which are successes, and $N-K$ failures) *without* replacement. If we did this with replacement, then this scenario would be represented as $\text{Bin}\left(n, \frac{K}{N}\right)$.

## 4 Continuous Random Variables

### 4.1 Continuous Random Variables Basics

**Probability Density Function (PDF):** The **probability density function (PDF)** of a continuous RV $X$ is the function $f_X : \mathbb{R} \to \mathbb{R}$, such that the following properties hold:

- $f_X(z) \ge 0$ for all $z \in \mathbb{R}$
- $\int_{-\infty}^{\infty} f_X(t)\, dt = 1$
- $\mathbb{P}(a \le X \le b) = \int_a^b f_X(w)\, dw$

**Cumulative Distribution Function (CDF):** The **cumulative distribution function (CDF)** of ANY random variable (discrete or continuous) is defined to be the function $F_X : \mathbb{R} \to \mathbb{R}$ with $F_X(t) = \mathbb{P}(X \le t)$. If $X$ is a *continuous* RV, we have:

$E(X_i) = 0.3^4 \, 0.2^2$

ATTC
0.3 0.3 0.3 0.2

## More practice with linearity of expectation

ATTGGGCGTAGAATGTC

A DNA sequence can be thought of as a string made up of 4 bases:

A, T, G, C

Suppose that the DNA sequence is random: the base in each position is selected independently of other positions, and for each particular position, one of the 4 bases is selected such that the letters G and C occur with probability 0.2 each and A and T occur with probability 0.3 each.

| | |
|---|---|
| G | 0.2 |
| C | 0.2 |
| A | 0.3 |
| T | 0.3 |

**In a sequence of length n, what is the expected number of occurrences of the sequence AATGTC?**

$\boxed{X : \# \text{ of occurrences of AATGTC}}$

$\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$



$E(X) = ?$

$X_i = \begin{cases} 1 & \text{if } \sigma_i \sigma_{i+1} \sigma_{i+2} \sigma_{i+3} \sigma_{i+4} \sigma_{i+5} = AATGTC \\ 0 & \end{cases}$

otherwise

$X_5$

AATGAA

$i$ $\quad$ $i+5$ $\quad$ $n-5$ $\quad$ $n-1$ $\quad$ $n$

$$X = X_1 + X_2 + X_3 + \dots X_5 \dots X_9 + X_{n-5}$$

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_{n-5})$$

AATGTC

~~AATGTC~~

$$= \underline{(n-5)}\, 0.3^4\, 0.2^2$$

## Example: Returning Homeworks

- Class with $n$ students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

| $\mathbf{Pr}(\boldsymbol{\omega})$ | $\boldsymbol{\omega}$ | $\boldsymbol{X}(\boldsymbol{\omega})$ |
|---|---|---|
| $1/6$ | $1,2,3$ | $3$ |
| $1/6$ | $1,3,2$ | $1$ |
| $1/6$ | $2,1,3$ | $1$ |
| $1/6$ | $2,3,1$ | $0$ |
| $1/6$ | $3,1,2$ | $0$ |
| $1/6$ | $3,2,1$ | $1$ |

Decompose: What is $X_i$?

$\boxed{X_i = 1 \text{ iff } i^{th} \text{ student gets own HW back}}$

LOE: $\boxed{X = X_1 + X_2 + \cdots + X_n}$

So $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: $\mathbb{E}[X_i] = \frac{1}{n}$

Therefore, $\mathbb{E}[X] = n \cdot \frac{1}{n} = 1$

## Example: Returning Homeworks

- Class with $n$ students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

What is $\mathbb{E}[X^2]$?

$E(X_i) = \frac{1}{n}$

$X_i = 1$ iff $i^{th}$ student gets own HW back

$X = X_1 + X_2 + \cdots + X_n$

$$= \sum_{k=0}^{n} k^2 \, P(X=k)$$

$$E(X^2) = E\left((X_1 + X_2 + \cdots + X_n)(X_1 + X_2 + \cdots + X_n)\right)$$

$$= E\left[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} X_i X_j\right]$$

$$= \sum_{i=1} E(X_i^2) + \sum_{i=1} \sum_{\substack{j=1 \\ j \neq i}} E(X_i X_j)$$

$$E(X_i) = \frac{1}{n}$$

$$\frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$$= n \cdot \frac{1}{n} + n(n-1) \frac{1}{n(n-1)}$$

$$= 2$$