Section 6 – Solutions

Review

1) **Cumulative Distribution Function (cdf)**: For any random variable (discrete or continuous) X, the cumulative distribution function is defined as

$$F_X(x) = \mathbb{P}(X \leqslant x).$$

Notice that this function must be monotonically nondecreasing: if $x < y$ then $F_X(x) \leqslant F_X(y)$, because $\mathbb{P}(X \leqslant x) \leqslant \mathbb{P}(X \leqslant y)$. Also notice that since probabilities are between 0 and 1, that $0 \leqslant F_X(x) \leqslant 1$ for all x, with $\lim_{x\to-\infty} F_X(x) = 0$ and $\lim_{x\to+\infty} F_X(x) = 1$.

2) **Continuous Random Variable**: A continuous random variable X is one for which its cumulative distribution function $F_X(x) : \mathbb{R} \to \mathbb{R}$ is continuous everywhere. A continuous random variable has an uncountably infinite number of values.

3) **Probability Density Function (pdf or density)**: Let X be a continuous random variable. Then the probability density function $f_X(x) : \mathbb{R} \to \mathbb{R}$ of X is defined as $f_X(x) = \frac{d}{dx} F_X(x)$. Turning this around, it means that $F_X(x) = \mathbb{P}(X \leqslant x) = \int_{-\infty}^{x} f_X(t)\, dt$. From this, it follows that $\mathbb{P}(a \leqslant X \leqslant b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$ and that $\int_{-\infty}^{\infty} f_X(x)dx = 1$. From the fact that $F_X(x)$ is monotonically nondecreasing it follows that $f_X(x) \geqslant 0$ for every real number x.

   If X is a continuous random variable, note that in general $f_X(a) \neq \mathbb{P}(X = a)$, since $\mathbb{P}(X = a) = F_X(a) - F_X(a) = 0$ for all a. However, the probability that X is close to a is proportional to $f_X(a)$: for small δ, $\mathbb{P}\left(a - \frac{\delta}{2} < X < a + \frac{\delta}{2}\right) \approx \delta f_X(a)$.

4) **i.i.d. (independent and identically distributed)**: Random variables $X_1, \ldots, X_n$ are i.i.d. (or iid) if they are independent and have the same probability mass function or probability density function.

5) **Discrete to Continuous:**

	Discrete	**Continuous**
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leqslant x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[X] = \sum_x x p_X(x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

6) **Uniform:** $X \sim \text{Uniform}(a, b)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

   $\mathbb{E}[X] = \frac{a+b}{2}$ and $\text{Var}(X) = \frac{(b-a)^2}{12}$. This represents each real number from $[a, b]$ to be equally likely.

7) **Exponential:** $X \sim \text{Exponential}(\lambda)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

   $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$. $F_X(x) = 1 - e^{-\lambda x}$ for $x \geqslant 0$. The exponential random variable is the continuous analog of the geometric random variable: it represents the waiting time to the next event, where $\lambda > 0$ is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential random variable X is memoryless:

$$\text{for any } s, t \geqslant 0,\ \mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

The geometric random variable also has this property.

8) Normal (Gaussian, "bell curve"): $X \sim \mathcal{N}(\mu,\ \sigma^2)$ iff X has the following probability density function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \mathbb{R}$$

$\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$. The "standard normal" random variable is typically denoted Z and has mean 0 and variance 1: if $X \sim \mathcal{N}(\mu,\ \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. The CDF has no closed form, but we denote the CDF of the standard normal as $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leqslant z)$. Note from symmetry of the probability density function about $z = 0$ that: $\Phi(-z) = 1 - \Phi(z)$.

Task 1 – Continuous r.v. example

Suppose that X is a random variable with pdf

$$f_X(x) = \begin{cases} 2C(2x - x^2) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

where C is an appropriately chosen constant.

a) What must the constant C be for this to be a valid pdf?

For $f_X(x)$ to be a valid PDF, $f_X(x)$ must be non-negative and the area under the graph must be 1. For $0 \leqslant x \leqslant 2$, we have $2x - x^2 = x(2 - x) \geqslant 0$ so we only need $C \geqslant 0$ for f_X to be non-negative everywhere. Computing the area under the graph as a function of C gives us

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^2 2C(2x - x^2)dx = 2C\left(x^2 - \frac{1}{3}x^3\Big|_0^2\right) = 2C\frac{4}{3} = \frac{8}{3}C$$

Setting this equation equal to 1, and solving for C gives use $C = \frac{3}{8}$.

b) Using this C, what is $\mathbb{P}(X > 1)$?

The $\mathbb{P}(X > 1) = \int_1^\infty f_X(t)dt$. Using our value for C that we found in the previous part we can compute this integral as follows:

$$\int_1^\infty f_X(t)dt = \int_1^2 \frac{6}{8}\left(2x - x^2\right)dt = \frac{6}{8}\left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2 = \frac{1}{2}$$

Alternatively, $\mathbb{P}(X > 1) = 1 - \mathbb{P}(X \leqslant 1) = 1 - F_X(1) = 1 - \int_{-\infty}^1 f_X(t)dt$. Using our value for C that we found in the previous part we can compute this integral as follows:

$$\int_{-\infty}^1 f_X(t)dt = \int_0^1 \frac{6}{8}\left(2x - x^2\right)dt = \frac{6}{8}\left(x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2}$$

Plugging this value into our initial equation gives $P(X > 1) = 1 - \frac{1}{2} = \frac{1}{2}$.

Task 2 – Will the battery last?

The owner of a car starts on a 5000 mile road trip. Suppose that the number of miles that the car will run before its battery wears out is exponentially distributed with expectation 10,000 miles. After successfully driving for 2000 miles on the trip without the battery wearing out, what is the probability that she will be able to complete the trip without replacing the battery?

Let N be a r.v. denoting the number of miles until the battery wears out. Then $N \sim \exp(10,000^{-1})$, because N measures the "time" (in this case miles) before an occurrence (the battery wears out) with expectation 10,000. Since this is an exponential distribution, and the expectation of an exponential distribution is $\frac{1}{\lambda}$, $\lambda = \frac{1}{10,000}$. Therefore, via the property of memorylessness of the exponential distribution:

$$\mathbb{P}(N \geqslant 5000 \mid N \geqslant 2000) = \mathbb{P}(N \geqslant 3000) = 1 - \mathbb{P}(N \leqslant 3000) = 1 - \left(1 - e^{-\frac{3000}{10000}}\right) \approx 0.741$$

Task 3 – Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1]$ and uniform on $[1, 2]$, but

$$\mathbb{P}(1 \leqslant X \leqslant 2) = 2 \cdot \mathbb{P}(0 \leqslant X < 1).$$

Outside of $[0, 2]$ the density is 0. What is the PDF and CDF of X?

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c & 0 < x \leqslant 1 \\ d & 1 < x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$. The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

Additionally,

$$d = F_X(2) - F_X(1) = \mathbb{P}(1 \leqslant X \leqslant 2) = 2 \cdot \mathbb{P}(0 \leqslant X \leqslant 1) = 2 \cdot (F_X(1) - F_X(0)) = 2c$$

To solve for c and d in our PDF, we need only solve the system of two equations from above: $d + c = 1$, $d = 2c$. So, $d = \frac{2}{3}$ and $c = \frac{1}{3}$. Taking the integral of the PDF yields the CDF, which looks like

$$F_X(x) = \begin{cases} 0 & x \leqslant 0 \\ \frac{1}{3}x & 0 < x \leqslant 1 \\ \frac{2}{3}x - \frac{1}{3} & 1 < x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

Another approach: Taking the integral of each component yields the CDF, which looks like

$$F_X(x) = \begin{cases} 0 & x \leqslant 0 \\ cx & 0 < x \leqslant 1 \\ d(x-1) + c & 1 < x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

To solve for c and d, we use the provided condition that $\mathbb{P}(1 \leqslant X \leqslant 2) = 2 \cdot \mathbb{P}(0 \leqslant X < 1)$, which implies that

$$F_X(2) - F_X(1) = 2 \cdot (F_X(1) - F_X(0))$$

Plugging in the values of our CDF for the values of x gives

$$d(2-1) + c - c = 2(c - 0)$$
$$d = 2c$$

By considering the area under the PDF, which must sum to 1.

$$c(1-0) + d(2-1) = 1$$

$$c + d = 1$$

From these 2 equations $d = 2c$ and $c + d = 1$, we can solve and get $c = \frac{1}{3}$ and $d = \frac{2}{3}$.

Task 4 – Max of uniforms

Let $U_1, U_2, \ldots, U_n$ be mutually independent Uniform random variables on $(0,1)$. Find the CDF and PMF for the random variable $Z = \max(U_1, \ldots, U_n)$.

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, ..., a_n)$ is less than some constant c, if and only if each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

$$\begin{aligned}
F_Z(x) = \mathbb{P}(Z \leqslant x) &= \mathbb{P}(\max(U_1, ..., U_n) \leqslant x) \\
&= \mathbb{P}(U_1 \leqslant x, ..., U_n \leqslant x) \\
&= \mathbb{P}(U_1 \leqslant x) \cdot ... \cdot \mathbb{P}(U_n \leqslant x) && \text{[independence]} \\
&= F_{U_1}(x) \cdot ... \cdot F_{U_n}(x) \\
&= F_U(x)^n && \text{[where } U \sim \text{Unif}(0,1)\text{]}
\end{aligned}$$

So the CDF of Z is

$$F_Z(x) = \begin{cases} 0 & x < 0 \\ x^n & 0 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

To find the PDF, we take the derivative of each part of the CDF, which gives us the following

$$f_Z(x) = \begin{cases} n\, x^{n-1} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Task 5 – Grading on a curve

In some classes (not CSE classes) an examination is regarded as being good (in the sense of determining a valid spread for those taking it) if the test scores of those taking it are well approximated by a normal density function. The instructor often uses the test scores to estimate the normal parameters μ and σ^2 and then assigns a letter grade of A to those whose test score is greater than $\mu + \sigma$, B to those whose score is between μ and $\mu + \sigma$, C to those whose score is between $\mu - \sigma$ and μ, D to those whose score is between $\mu - 2\sigma$ and $\mu - \sigma$ and F to those getting a score below $\mu - 2\sigma$. If the instructor does this and a student's grade on the test really is normally distributed with mean μ and variance σ^2, what is the probability that student will get each of the possible grades A,B,C,D and F? (Use a table for anything you can't calculate.)

We can solve for each of these probabilities by standardizing the normal curve and then looking up each bound in the Z-table. Let X be the students score on the test. Then we have

$$\mathbb{P}(A) = \mathbb{P}(X \geqslant \mu + \sigma) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \geqslant 1\right) = 1 - \mathbb{P}\left(\frac{X-\mu}{\sigma} < 1\right)$$

By the closure properties of the normal random variable, $\frac{X-\mu}{\sigma}$ is distributed as a normal random variable with mean 0 and variance 1. Since this is the standard normal, we can plug it into our Φ-table to get the following:

$$\mathbb{P}(A) = 1 - \Phi(1) = 1 - 0.84134 = 0.15866$$

The other probabilities can be found using a similar approach:

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}(\mu < X < \mu + \sigma) = \Phi(1) - \Phi(0) = 0.34134 \\
\mathbb{P}(C) &= \mathbb{P}(\mu - \sigma < X < \mu) = \Phi(0) - \Phi(-1) = 0.34134 \\
\mathbb{P}(D) &= \mathbb{P}(\mu - 2\sigma < X < \mu - \sigma) = \Phi(-1) - \Phi(-2) = 0.13591 \\
\mathbb{P}(F) &= \mathbb{P}(X < \mu - 2\sigma) = \Phi(-2) = 0.02275
\end{aligned}$$

Task 6 – Throwing a dart

Consider the closed unit circle of radius r, i.e., $S = \{(x, y) : x^2 + y^2 \leqslant r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S. Concretely this means that the probability that the dart lands in any particular area of size A (that is entirely inside the circle of radius R), is equal to $\frac{\text{A}}{\text{Area of whole circle}}$. The density outside the circle of radius r is 0.

Let X be the distance the dart lands from the center. What is the CDF and pdf of X? What is $\mathbb{E}[X]$ and $\mathrm{Var}(X)$?

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x, that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\pi x^2/\pi r^2$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{r^2} & 0 < x \leqslant r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) = \begin{cases} \frac{2x}{r^2} & 0 < x \leqslant r \\ 0 & \text{otherwise} \end{cases}$$

Using the definition of expectation we get

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_0^r x\frac{2x}{r^2}\,\mathrm{d}x = \frac{2}{3r^2}\left(x^3\Big|_0^r\right) = \frac{2}{3}r$$

We know that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,\mathrm{d}x = \int_0^r x^2\frac{2x}{r^2}\,\mathrm{d}x = \frac{2}{4r^2}\left(x^4\Big|_0^r\right) = \frac{1}{2}r^2$$

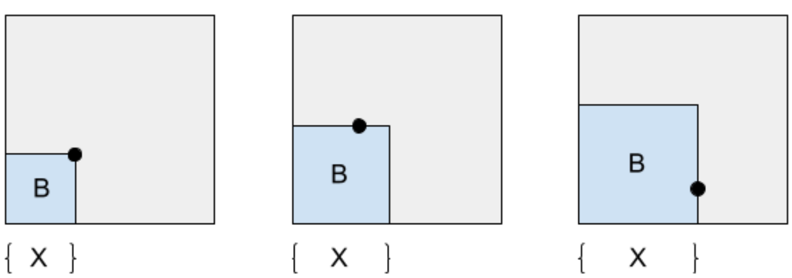
Plugging this into our variance equation gives

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{2}r^2 - \left(\frac{2}{3}r\right)^2 = \frac{1}{18}r^2$$

Task 7 – A square dartboard?

You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest *square* B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B. For random variable X, find the CDF, PDF, $\mathbb{E}[X]$, and $\mathrm{Var}(X)$.

See the image below for three examples of how X can take on a value.



Since $F_X(x)$ is the probability that the dart lands inside the square of side length x, that probability is the area of a square of length x divided by the area of the square of length radius s (i.e., x^2/r^2). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/s^2, & \text{if } 0 \leqslant x \leqslant s \\ 1, & \text{if } x > s \end{cases}$$

To find the PDF, we just need to take the derivative of the CDF, which gives us the following:

$$f_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_X(x) = \begin{cases} 2x/s^2, & \text{if } 0 \leqslant x \leqslant s \\ 0, & \text{otherwise} \end{cases}$$

Using the definition of expectation and variance we can compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ in the following manner:

$$\mathbb{E}[X] = \int_0^s x f_X(x)\,\mathrm{d}x = \int_0^s \frac{2x^2}{s^2}\,\mathrm{d}x = \frac{2}{s^2}\int_0^s x^2\,\mathrm{d}x = \frac{2}{3s^2}\left[x^3\right]_0^s = \frac{2}{3}s$$

$$\mathbb{E}[X^2] = \int_0^s x^2 f_X(x)\,\mathrm{d}x = \int_0^s \frac{2x^3}{s^2}\,\mathrm{d}x = \frac{2}{s^2}\int_0^s x^3\,\mathrm{d}x = \frac{1}{2s^2}\left[x^4\right]_0^s = \frac{1}{2}s^2$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2}s^2 - \left(\frac{2}{3}s\right)^2 = \frac{1}{18}s^2$$

Task 8 – Normal questions at the table

a) Let X be a normal random with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute $\mathbb{P}(4 < X < 16)$.

Let $\frac{X-10}{6} = Z$. By the scale and shift properties of normal random variables $Z \sim \mathcal{N}(0,1)$.

$$\mathbb{P}(4 < X < 16) = \mathbb{P}\left(\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\right) = \mathbb{P}(-1 < Z < 1)$$

$$= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$$

b) Let X be a normal random variable with mean 5. If $\mathbb{P}(X > 9) = 0.2$, approximately what is $\mathrm{Var}(X)$?

Let $\sigma^2 = \mathrm{Var}(X)$. Then,

$$\mathbb{P}(X > 9) = \mathbb{P}\left(\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\right) = 1 - \Phi\left(\frac{4}{\sigma}\right) = 0.2$$

So, $\Phi\left(\frac{4}{\sigma}\right) = 0.8$. Looking up the phi values in reverse lets us undo the Φ function, and gives us $\frac{4}{\sigma} = 0.845$. Solving for σ we get $\sigma \approx 4.73$, which means that the variance is about 22.4.

c) Let X be a normal random variable with mean 12 and variance 4.
Find the value of c such that $\mathbb{P}(X > c) = 0.10$.

$$\mathbb{P}(X > c) = \mathbb{P}\left(\frac{X-12}{2} > \frac{c-12}{2}\right) = 1 - \Phi\left(\frac{c-12}{2}\right) = 0.1$$

So, $\Phi\left(\frac{c-12}{2}\right) = 0.9$. Looking up the phi values in reverse lets us undo the Φ function, and gives us $\frac{c-12}{2} = 1.29$. Solving for c we get $c \approx 14.58$.

Task 9 – Batteries and exponential distributions

Let X_1, X_2 be independent exponential random variables, where X_i has parameter λ_i, for $1 \leqslant i \leqslant 2$. Let $Y = \min(X_1, X_2)$.

a) Show that Y is an exponential random variable with parameter $\lambda = \lambda_1 + \lambda_2$. Hint: Start by computing $\mathbb{P}(Y > y)$. Two random variables with the same CDF have the same pdf. Why?

We start with computing $\mathbb{P}(Y > y)$, by substituting in the definition of Y.

$$\mathbb{P}(Y > y) = \mathbb{P}(\min\{X_1, X_2\} > y)$$

The probability that the minimum of two values is above a value is the chance that both of them are above that value. From there, we can separate them further because X_1 and X_2 are independent.

$$\mathbb{P}(X_1 > y \cap X_2 > y) = \mathbb{P}(X_1 > y)\mathbb{P}(X_2 > y) = e^{-\lambda_1 y}e^{-\lambda_2 y}$$
$$= e^{-(\lambda_1+\lambda_2)y} = e^{-\lambda y}$$

So $F_Y(y) = 1 - \mathbb{P}(Y > y) = 1 - e^{-\lambda y}$ and $f_Y(y) = \lambda e^{-\lambda y}$ so $Y \sim \text{Exp}(\lambda)$, since this is the same CDF and PDF as an exponential distribution with parameter $\lambda = \lambda_1 + \lambda_2$.

b) What is $\mathbb{P}(X_1 < X_2)$? (Use the law of total probability.)

By the law of total probability,

$$\mathbb{P}(X_1 < X_2) = \int_0^\infty \mathbb{P}(X_1 < X_2 \mid X_1 = x) f_{X_1}(x)\,\mathrm{d}x = \int_0^\infty \mathbb{P}(X_2 > x)\lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x =$$

$$\int_0^\infty e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}\,\mathrm{d}x = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

c) You have a digital camera that requires two batteries to operate. You purchase n batteries, labelled $1, 2, \ldots, n$, each of which has a lifetime that is exponentially distributed with parameter λ, independently of all other batteries. Initially, you install batteries 1 and 2. Each time a battery fails, you replace it with the lowest-numbered unused battery. At the end of this process, you will be left with just one working battery. What is the expected total time until the end of the process? Justify your answer.

Let T be the time until the end of the process. We are trying to find $\mathbb{E}[T]$. $T = Y_1 + \ldots + Y_{n-1}$ where Y_i is the time until we have to replace a battery from the ith pair. The reason it there are only $n-1$ RVs in the sum is because there are $n-1$ times where we have two batteries and wait for one to fail. By part (a), the time for one to fail is the min of exponentials, so $Y_i \sim$ Exponential(2λ). Hence the expected time for the first battery to fail is $\frac{1}{2\lambda}$. By linearity and memorylessness, $\mathbb{E}[T] = \sum_{i=1}^{n-1}\mathbb{E}[Y_1] = \frac{n-1}{2\lambda}$.

d) In the scenario of the previous part, what is the probability that battery i is the last remaining battery as a function of i? (You might want to use the memoryless property of the exponential distribution that has been discussed.)

If there are two batteries i, j in the flashlight, by part (b), the probability each outlasts each other is $1/2$. Hence, the last battery n has probability $1/2$ of being the last one remaining. The second to last battery $n-1$ has to beat out the previous battery and the n^{th}, so the probability it lasts the longest is $(1/2)^2 = 1/4$. Work down inductively to get that the probability the i^{th} is the last remaining is $(1/2)^{n-i+1}$ for $i \geqslant 3$. Finally the first two batteries share the remaining probability as they start at the same time, with probability $(1/2)^{n-1}$ each.