# CSE 312 Section 8

## Tail Bounds, Joint Distributions, and the Law of Total Expectation

# Administrivia

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- HW5
  - due Tuesday, August 1st
  - Late deadline today
- HW6
  - Released on the course website
  - Due next Tuesday, August 8th

# Review

# Review

- **Multivariate: Discrete to Continuous:**

| | **Discrete** | **Continuous** |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$ | $f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$ |
| **Joint range/support** $\Omega_{X,Y}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \leq x, s \leq y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$ |
| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$ |
| **Expectation** | $\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$ | $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$ |
| **Independence** must have | $\forall x,y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ | $\forall x,y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ |

# Review (cont.)

- **Law of Total Probability (r.v. version):** If $X$ is a discrete random variable, then

$$\mathrm{P(A)} = \sum_{x \in \Omega_X} P(A|X = x)p_X(x)$$

($X$ is discrete.)

- **Law of Total Expectation (event version):** Let $X$ be a discrete random variable, and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$\mathrm{E[X]} = \sum_{i=1}^{n} E[X|A_i]P(A_i)$$

- **Law of Total Expectation (r.v. version):** Suppose $X$ and $Y$ are random variables. Then,

$$E[X] = \sum_{y} E[X|Y = y]p_Y(y)$$

($Y$ is discrete.)

# Review (cont.)

- Conditional Distributions (discrete case): $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$. The continuous version is the same, but with a pdf instead of a pmf.
- Conditional Expectation (discrete case): $E[X|Y = y] = \sum_x x p_{X|Y}(x|y)$. The continuous version is the same, but with a pdf instead of a pmf (and integral instead of sum).
  - *Linearity of expectation still applies:* $E[X + Y|A] = E[X|A] + E[Y|A]$

- **Continuous Law of Total Probability:**

$$\mathbb{P}(A) = \int_{x \in \Omega_X} \mathbb{P}(A|X = x) f_X(x) dx$$

- **Continuous Law of Total Expectation:**

$$\mathbb{E}[X] = \int_{y \in \Omega_Y} \mathbb{E}[X|Y = y] f_Y(y) dy$$

# Review (cont.)

- **Markov's Inequality:** Let $X$ be a nonnegative random variable and $\alpha > 0$. Then
$$\mathrm{P}(\mathrm{X} \geq \alpha) \leq \frac{E[X]}{\alpha}$$
- **Chebyshev's Inequality:** Let $Y$ be a random variable with $E[Y] = \mu$ and $Var(Y) = \sigma^2$. Then, for any $\alpha > 0$,
$$P(|Y - \mu| \geq \alpha) \leq \frac{\sigma^2}{\alpha^2}$$
- **(Multiplicative) Chernoff Bound:** Suppose $X \sim Bin(n, p)$ and $\mu = np$. Then for any $0 < \delta < 1$,
$$\mathrm{P}(\mathrm{X} \geq (1+\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}}$$
$$P(X \leq (1-\delta)\mu) \leq e^{-\frac{\delta^2 \mu}{2}}$$
- **Extra tidbit:** Other useful inequalities in probability include the **union bound** and **Hoeffding's Inequality** (in case you wanted to learn more). There are tons of them.

# Problem 3 – Exponential Tail Bound

# Problem 3

- Let $X \sim Exp(\lambda)$ and $k > \frac{1}{\lambda}$.
  - (a) Use Markov's inequality to bound $P(X \geq k)$.
  - (b) Use Chebyshev's inequality to bound $P(X \geq k)$.
  - (c) What is the exact formula for $P(X \geq k)$?
  - (d) For $\lambda k \geq 3$, how do the bounds given in (a), (b), and (c) compare?

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$$P(X \geq k) = P\left(X - \frac{1}{\lambda} \geq k - \frac{1}{\lambda}\right) \leq P\left(\left|X - \frac{1}{\lambda}\right| \geq k - \frac{1}{\lambda}\right) \leq \frac{1}{\lambda^2 \left(k - \frac{1}{\lambda}\right)^2} = \frac{1}{(\lambda k - 1)^2}$$

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- (c) Take the complement of the CDF: $P(X \geq k) = e^{-\lambda k}$
- (d) $e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$, so Markov's inequality gives the worst bound.

# Problem 7 – Continuous joint density

# Problem 7

The joint density of $X$ and $Y$ is given by

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

and the joint density of $W$ and $V$ is given by

$$f_{W,V}(w,v) = \begin{cases} 2 & 0 < w < v, 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Are $X$ and $Y$ independent? Are $W$ and $V$ independent?

# Problem 7 Solution

- For two random variables $X, Y$ to be independent, we need $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ for all $x \in \Omega_X$. First, we marginalize:

$$\int_0^\infty xe^{-(x+y)}dy = -xe^{-(x+y)}\Big|_0^\infty = e^{-x}x$$

Same for $Y$:

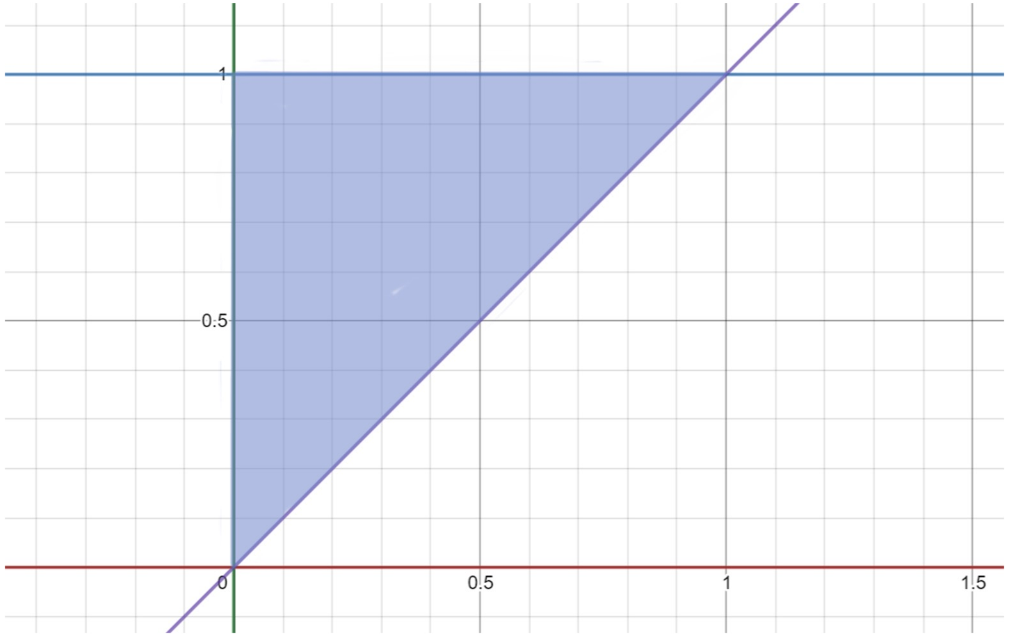
$$f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}$$

(I believe you use $u = -x - y$ as a substitution to solve this integral.)

Since $e^{-x}x \cdot e^{-y} = xe^{-x-y} = xe^{-(x+y)}$ for all $x, y > 0$, $X$ and $Y$ are independent.

# Problem 7 Solution

- $W$ and $V$ are clearly not independent: $\Omega_W = (0, 1)$ and $\Omega_V = (0, 1)$, but $\Omega_{W,V}$ is not equal to their Cartesian product (it'd be a rectangle if it were).



# Problem 9 – Lemonade Stand

# Problem 9

- Suppose I run a lemonade stand that costs me \$100 a day to operate. I sell a single lemonade for \$20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, $n_1$ people walk by my stand, and each buys a drink independently with probability $p_1$. If it isn't raining, $n_2$ people walk by my stand, and each buys a drink independent with probability $p_2$. It rains each day with probability $p_{rain}$ independently of every other day. Let $X$ be my profit over the next week. In terms of $n_1, n_2, p_1, p_2, p_{rain}$, what is $E[X]$?

# Problem 9 Solution

- Let $R$ be the event it rains. Let $X_i$ be how many drinks I sell on day $i$ for $i = 1, \dots, 7$. We are interested in $X = \sum_{i=1}^{7}(20X_i - 100)$.

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  - We also know that $X_i|R \sim Bin(n_1, p_1) \Longrightarrow E[X_i|R] = n_1 p_1$ from the problem statement.
  - Similarly, $X_i|R^C \sim Bin(n_2, p_2) \Longrightarrow E[X_i|R^C] = n_2 p_2$

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  We then apply the Law of Total Expectation:
  $\mu_{X_i} = E[X_i] = E[X_i|R]P(R) + E[X_i|R^C]P(R^C) = n_1 p_1 \cdot p_{rain} + n_2 p_2 \cdot (1 - p_{rain})$

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  Then Linearity of Expectation:

$$E[X] = E\left[\sum_{i=1}^{7}(20X_i - 100)\right] = 20\left(\sum_{i=1}^{7} E[X_i]\right) - 700$$
$$= 140\left(n_1 p_1 \cdot p_{rain} + n_2 p_2 \cdot (1 - p_{rain})\right) - 700$$

# Problem 8 – Trapped Miner

# Problem 8

- A miner is trapped in a mine containing 3 doors:
  - $D_1$: The 1st door leads to a tunnel that will take them to safety after 3 hours.
  - $D_2$: The 2nd door leads to a tunnel that will return them to the mine after 5 hours.
  - $D_3$: The 3rd door leads to a tunnel that returns them to the mine after a number of hours that is binomially distributed with parameters $(12, \frac{1}{3})$.
- At all times, they are equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

# Problem 8 Solution

- Let $T$ be the number of hours for the miner to reach safety. Let $D_i$ be the event the i$^{th}$ door is chosen. Finally let $T_3$ be the time it takes to return to the mine in the third case only (since it is a binomial r.v.).
  - $E[T_3] = 12 \cdot \frac{1}{3} = 4$ because we know the expectation of a binomial r.v.

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  - $E[T_3] = 12 \cdot \frac{1}{3} = 4$ because we know the expectation of a binomial r.v.
  - By Law of Total Expectation, linearity of expectation, and applying the conditional expectations given by the problem statement,

$$E[T] = E[T|D_1]P(D_1) + E[T|D_2]P(D_2) + E[T|D_3]P(D_3)$$

$$E[T] = 3 \cdot \frac{1}{3} + (5 + E[T]) \cdot \frac{1}{3} + (E[T_3 + T]) \cdot \frac{1}{3}$$

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Solve for $E[T]$ and we get $\boxed{E[T] = 12}$