

Section 5: Solutions

Review of Main Concepts

- **Cumulative Distribution Function (cdf):** For any random variable (discrete or continuous) X , the cumulative distribution function is defined as $F_X(x) = \mathbb{P}(X \leq x)$. Notice that this function must be monotonically nondecreasing: if $x < y$ then $F_X(x) \leq F_X(y)$, because $\mathbb{P}(X \leq x) \leq \mathbb{P}(X \leq y)$. Also notice that since probabilities are between 0 and 1, that $0 \leq F_X(x) \leq 1$ for all x , with $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow +\infty} F_X(x) = 1$.
- **Continuous Random Variable:** A continuous random variable X is one for which its cumulative distribution function $F_X(x) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous everywhere. A continuous random variable has an uncountably infinite number of values.
- **Probability Density Function (pdf or density):** Let X be a continuous random variable. Then the probability density function $f_X(x) : \mathbb{R} \rightarrow \mathbb{R}$ of X is defined as $f_X(x) = \frac{d}{dx} F_X(x)$. Turning this around, it means that $F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^x f_X(t) dt$. From this, it follows that $\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx$ and that $\int_{-\infty}^{\infty} f_X(x) dx = 1$. From the fact that $F_X(x)$ is monotonically nondecreasing it follows that $f_X(x) \geq 0$ for every real number x .

If X is a continuous random variable, note that in general $f_X(a) \neq \mathbb{P}(X = a)$, since $\mathbb{P}(X = a) = F_X(a) - F_X(a) = 0$ for all a . However, the probability that X is close to a is proportional to $f_X(a)$: for small δ , $\mathbb{P}(a - \frac{\delta}{2} < X < a + \frac{\delta}{2}) \approx \delta f_X(a)$.

- **i.i.d. (independent and identically distributed):** Random variables $X_1, \dots, X_n$ are i.i.d. (or iid) if they are independent and have the same probability mass function or probability density function.
- **Discrete to Continuous:**

	Discrete	Continuous
PMF/PDF	$p_X(x) = \mathbb{P}(X = x)$	$f_X(x) \neq \mathbb{P}(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
Expectation	$\mathbb{E}[X] = \sum_x x p_X(x)$	$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$
LOTUS	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

- **Standardizing:** Let X be any random variable (discrete or continuous, not necessarily normal), with $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$. If we let $Y = \frac{X - \mu}{\sigma}$, then $\mathbb{E}[Y] = 0$ and $\text{Var}(Y) = 1$.
- **Closure of the Normal Distribution:** Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$. That is, linear transformations of normal random variables are still normal.
- **“Reproductive” Property of Normals:** Let $X_1, \dots, X_n$ be independent normal random variables with $\mathbb{E}[X_i] = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$. Let $a_1, \dots, a_n \in \mathbb{R}$ and $b \in \mathbb{R}$. Then,

$$X = \sum_{i=1}^n (a_i X_i + b) \sim \mathcal{N}\left(\sum_{i=1}^n (a_i \mu_i + b), \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

There’s nothing special about the parameters – the important result here is that the resulting random variable is still normally distributed.

- **Law of Total Probability (Continuous):** A is an event, and X is a continuous random variable with density function $f_X(x)$.

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X = x) f_X(x) dx$$

- **Central Limit Theorem (CLT):** Let $X_1, \dots, X_n$ be iid random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2$. Let $X = \sum_{i=1}^n X_i$, which has $\mathbb{E}[X] = n\mu$ and $\text{Var}(X) = n\sigma^2$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, which has $\mathbb{E}[\bar{X}] = \mu$

and $Var(\bar{X}) = \frac{\sigma^2}{n}$. $\bar{X}$ is called the *sample mean*. Then, as $n \rightarrow \infty$, $\bar{X}$ approaches the normal distribution $\mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$. Standardizing, this is equivalent to $Y = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ approaching $\mathcal{N}(0, 1)$. Similarly, as $n \rightarrow \infty$, X approaches $\mathcal{N}(n\mu, n\sigma^2)$ and $Y' = \frac{X-n\mu}{\sigma\sqrt{n}}$ approaches $\mathcal{N}(0, 1)$.

It is no surprise that $\bar{X}$ has mean μ and variance σ^2/n – this can be done with simple calculations. The importance of the CLT is that, for large n , regardless of what distribution X_i comes from, $\bar{X}$ is *approximately normally distributed with mean μ and variance σ^2/n* . Don't forget the continuity correction, only when $X_1, \dots, X_n$ are discrete random variables.

- Zoo of Continuous Random Variables

(a) **Uniform:** $X \sim \text{Uniform}(a, b)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a)^2}{12}$. This represents each real number from $[a, b]$ to be equally likely.

(b) **Exponential:** $X \sim \text{Exponential}(\lambda)$ iff X has the following probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$. $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. The exponential random variable is the continuous analog of the geometric random variable: it represents the waiting time to the next event, where $\lambda > 0$ is the average number of events per unit time. Note that the exponential measures how much time passes until the next event (any real number, continuous), whereas the Poisson measures how many events occur in a unit of time (nonnegative integer, discrete). The exponential random variable X is memoryless:

$$\text{for any } s, t \geq 0, \mathbb{P}(X > s+t \mid X > s) = \mathbb{P}(X > t)$$

The geometric random variable also has this property.

(c) **Normal (Gaussian, “bell curve”):** $X \sim \mathcal{N}(\mu, \sigma^2)$ iff X has the following probability density function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad x \in \mathbb{R}$$

$\mathbb{E}[X] = \mu$ and $Var(X) = \sigma^2$. The “standard normal” random variable is typically denoted Z and has mean 0 and variance 1: if $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$. The CDF has no closed form, but we denote the CDF of the standard normal as $\Phi(z) = F_Z(z) = \mathbb{P}(Z \leq z)$. Note from symmetry of the probability density function about $z = 0$ that: $\Phi(-z) = 1 - \Phi(z)$.

1. True or False? [From Section 4]

Identify the following statements as true or false (true means always true). Justify your answer.

(a) For any random variable X , we have $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$. **Solution:**

True, since $0 \leq Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$, since the squaring necessitates the result is non-negative.

(b) Let X, Y be random variables. Then, X and Y are independent if and only if $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. **Solution:**

False. The forward implication is true, but the reverse is not. For example, if $X \sim \text{Uniform}(-1, 1)$ (equally likely to be in $\{-1, 0, 1\}$), and $Y = X^2$, we have $\mathbb{E}[X] = 0$, so $\mathbb{E}[X]\mathbb{E}[Y] = 0$. However, since $X = X^3$ (why?), $\mathbb{E}[XY] = \mathbb{E}[X^3] = \mathbb{E}[X] = 0$, we have that $\mathbb{E}[X]\mathbb{E}[Y] = 0 = \mathbb{E}[XY]$. However, X and

Y are not independent; indeed, $\mathbb{P}(Y = 0|X = 0) = 1 \neq \frac{1}{3} = \mathbb{P}(Y = 0)$.

- (c) Let $X \sim \text{Binomial}(n, p)$ and $Y \sim \text{Binomial}(m, p)$ be independent. Then, $X + Y \sim \text{Binomial}(n + m, p)$.
Solution:

True. X is the sum of n independent Bernoulli trials, and Y is the sum of m . So $X + Y$ is the sum of $n + m$ independent Bernoulli trials, so $X + Y \sim \text{Binomial}(n + m, p)$.

- (d) Let $X_1, \dots, X_{n+1}$ be independent Bernoulli(p) random variables. Then, $\mathbb{E}[\sum_{i=1}^n X_i X_{i+1}] = np^2$. **Solution:**

True. Notice that $X_i X_{i+1}$ is also Bernoulli (only takes on 0 and 1), but is 1 iff both are 1, so $X_i X_{i+1} \sim \text{Bernoulli}(p^2)$. The statement holds by linearity, since $\mathbb{E}[X_i X_{i+1}] = p^2$.

- (e) Let $X_1, \dots, X_{n+1}$ be independent Bernoulli(p) random variables. Then, $Y = \sum_{i=1}^n X_i X_{i+1} \sim \text{Binomial}(n, p^2)$.
Solution:

False. They are all Bernoulli p^2 as determined in the previous part, but they are not independent. Indeed, $\mathbb{P}(X_1 X_2 = 1 | X_2 X_3 = 1) = \mathbb{P}(X_1 = 1) = p \neq p^2 = \mathbb{P}(X_1 X_2 = 1)$.

- (f) If $X \sim \text{Bernoulli}(p)$, then $nX \sim \text{Binomial}(n, p)$. **Solution:**

False. The range of X is $\{0, 1\}$, so the range of nX is $\{0, n\}$. nX cannot be $\text{Bin}(n, p)$, otherwise its range would be $\{0, 1, \dots, n\}$.

- (g) If $X \sim \text{Binomial}(n, p)$, then $\frac{X}{n} \sim \text{Bernoulli}(p)$. **Solution:**

False. Again, the range of X is $\{0, 1, \dots, n\}$, so the range of $\frac{X}{n}$ is $\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$. Hence it cannot be $\text{Ber}(p)$, otherwise its range would be $\{0, 1\}$.

- (h) For any two independent random variables X, Y , we have $\text{Var}(X - Y) = \text{Var}(X) - \text{Var}(Y)$. **Solution:**

False. $\text{Var}(X - Y) = \text{Var}(X + (-Y)) = \text{Var}(X) + (-1)^2 \text{Var}(Y) = \text{Var}(X) + \text{Var}(Y)$.

2. Fun with Poissons [From Section 4]

Let $X \sim \text{Poisson}(\lambda_1)$ and $Y \sim \text{Poisson}(\lambda_2)$, and X and Y are independent.

- (a) Show that $X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$ [This was done in class.] **Solution:**

To show that a random variable is distributed according to a particular distribution, we must show that

they have the same PMF. Thus, we are trying to show that $P(X + Y = n) = e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}$

$$\begin{aligned}
 P(X + Y = n) &= \sum_{k=0}^n P(X = k \cap Y = n - k) \\
 &= \sum_{k=0}^n P(X = k)P(Y = n - k) && \text{[X and Y are independent]} \\
 &= \sum_{k=0}^n e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} \\
 &= e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^n \frac{\lambda_1^k}{k!} \frac{\lambda_2^{n-k}}{(n-k)!} \\
 &= e^{-(\lambda_1 + \lambda_2)} \sum_{k=0}^n \frac{1}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \\
 &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \lambda_1^k \lambda_2^{n-k} \\
 &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{k=0}^n \binom{n}{k} \lambda_1^k \lambda_2^{n-k} \\
 &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n && \text{[Binomial Theorem]}
 \end{aligned}$$

(b) Show that $P(X = k \mid X + Y = n) = P(W = k)$ where $W \sim \text{Bin}(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})$ **Solution:**

$$\begin{aligned}
 P(X = k \mid X + Y = n) &= \frac{P(X = k \cap X + Y = n)}{P(X + Y = n)} \\
 &= \frac{P(X = k \cap Y = n - k)}{P(X + Y = n)} \\
 &= \frac{P(X = k)P(Y = n - k)}{P(X + Y = n)} && \text{[X and Y are independent]} \\
 &= \frac{e^{-\lambda_1} \frac{\lambda_1^k}{k!} \cdot e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1 + \lambda_2)} \frac{(\lambda_1 + \lambda_2)^n}{n!}} \\
 &= \frac{\frac{\lambda_1^k}{k!} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}}{\frac{(\lambda_1 + \lambda_2)^n}{n!}} \\
 &= \frac{n!}{k!(n-k)!} \cdot \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\
 &= \binom{n}{k} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^k (\lambda_1 + \lambda_2)^{n-k}} \\
 &= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-k} \\
 &= \binom{n}{k} p^k (1-p)^{n-k}, \text{ where } p = \frac{\lambda_1}{\lambda_1 + \lambda_2}
 \end{aligned}$$

3. Uniform2

Robbie decided he wanted to create a “new” type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We’ll denote a random variable X having the “Uniform-2” distribution as $X \sim \text{Uniform2}(a, b, c, d)$, where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

- (a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition). **Solution:**

$$f_X(x) = \begin{cases} \frac{1}{(b-a)+(d-c)}, & x \in [a, b] \cup [c, d] \\ 0, & \text{otherwise} \end{cases}$$

- (b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition). **Solution:**

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, a) \\ \frac{(x-a)}{(b-a)+(d-c)}, & x \in [a, b] \\ \frac{(b-a)}{(b-a)+(d-c)}, & x \in [b, c] \\ \frac{(b-a)+(x-c)}{(b-a)+(d-c)}, & x \in [c, d] \\ 1, & x \in [d, \infty) \end{cases}$$

4. Create the distribution

Suppose X is a continuous random variable that is uniform on $[0, 1]$ and uniform on $[1, 2]$, but

$$\mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X < 1).$$

Outside of $[0, 2]$ the density is 0. What is the PDF and CDF of X ? **Solution:**

The fact that X is uniform on each of the intervals means that its PDF is constant on each. So,

$$f_X(x) = \begin{cases} c & 0 < x \leq 1 \\ d & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $F_X(1) - F_X(0) = c$ and $F_X(2) - F_X(1) = d$. The area under the PDF must be 1, so

$$1 = F_X(2) - F_X(0) = F_X(2) - F_X(1) + F_X(1) - F_X(0) = d + c$$

Additionally,

$$d = F_X(2) - F_X(1) = \mathbb{P}(1 \leq X \leq 2) = 2 \cdot \mathbb{P}(0 \leq X \leq 1) = 2 \cdot (F_X(1) - F_X(0)) = 2c$$

To solve for c and d in our PDF, we need only solve the system of two equations from above: $d + c = 1$, $d = 2c$.

So, $d = \frac{2}{3}$ and $c = \frac{1}{3}$. Taking the integral of the PDF yields the CDF, which looks like

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{3}x & 0 < x \leq 1 \\ \frac{2}{3}x - \frac{1}{3} & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$

5. Max of uniforms

Let $U_1, U_2, \dots, U_n$ be mutually independent Uniform random variables on $(0, 1)$. Find the CDF and pmf for the random variable $Z = \max(U_1, \dots, U_n)$. **Solution:**

The key idea for solving this question is realizing that the max of n numbers $\max(a_1, \dots, a_n)$ is less than some constant c , if and only if each individual number is less than that constant c (i.e. $a_i < c$ for all i). Using this idea, we get

$$\begin{aligned} F_Z(x) &= \mathbb{P}(Z \leq x) = \mathbb{P}(\max(U_1, \dots, U_n) \leq x) \\ &= \mathbb{P}(U_1 \leq x, \dots, U_n \leq x) \\ &= \mathbb{P}(U_1 \leq x) \cdot \dots \cdot \mathbb{P}(U_n \leq x) && \text{[independence]} \\ &= F_{U_1}(x) \cdot \dots \cdot F_{U_n}(x) \\ &= F_U(x)^n && \text{[where } U \sim \text{Unif}(0, 1)] \end{aligned}$$

So the CDF of Z is

$$F_Z(x) = \begin{cases} 0 & x < 0 \\ x^n & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

To find the PDF, we take the derivative of each part of the CDF, which gives us the following

$$f_Z(x) = \begin{cases} n x^{n-1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

6. New PDF?

Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f_X(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.) **Solution:**

The area under the PDF is 1. So,

$$\int_0^\infty \frac{c}{1+x^2} dx = c \tan^{-1} x \Big|_0^\infty = c \left(\frac{\pi}{2} - 0 \right) = 1$$

Solving for c gives us $c = 2/\pi$. Using our value we found for c , and the definition of expectation we can compute $E[X]$ as follows:

$$\mathbb{E}[X] = \int_0^\infty \frac{cx}{1+x^2} dx = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} dx = \frac{1}{\pi} \ln(1+x^2) \Big|_0^\infty = \infty$$

7. Throwing a dart

Consider the closed unit circle of radius r , i.e., $S = \{(x, y) : x^2 + y^2 \leq r^2\}$. Suppose we throw a dart onto this circle and are guaranteed to hit it, but the dart is equally likely to land anywhere in S . Concretely this means that the probability that the dart lands in any particular area of size A (that is entirely inside the circle of radius R), is equal to $\frac{A}{\text{Area of whole circle}}$. The density outside the circle of radius r is 0.

Let X be the distance the dart lands from the center. What is the CDF and pdf of X ? What is $\mathbb{E}[X]$ and $\text{Var}(X)$?

Solution:

Since $F_X(x)$ is the probability that the dart lands inside the circle of radius x , that probability is the area of a circle of radius x divided by the area of the circle of radius r (i.e., $\pi x^2 / \pi r^2$). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x^2}{r^2} & 0 \leq x \leq r \\ 1 & x > r \end{cases}$$

To find the PDF we just need to take the derivative of the CDF, which give us the following:

$$f_X(x) = \begin{cases} \frac{2x}{r^2} & 0 < x \leq r \\ 0 & \text{otherwise} \end{cases}$$

Using the definition of expectation we get

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^r x \frac{2x}{r^2} dx = \frac{2}{3r^2} \left(x^3 \Big|_0^r \right) = \frac{2}{3}r$$

We know that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_0^r x^2 \frac{2x}{r^2} dx = \frac{2}{4r^2} \left(x^4 \Big|_0^r \right) = \frac{1}{2}r^2$$

Plugging this into our variance equation gives

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{2}r^2 - \left(\frac{2}{3}r \right)^2 = \frac{1}{18}r^2$$

8. A square dartboard?

You throw a dart at an $s \times s$ square dartboard. The goal of this game is to get the dart to land as close to the lower left corner of the dartboard as possible. However, your aim is such that the dart is equally likely to land at any point on the dartboard. Let random variable X be the length of the side of the smallest *square* B in the lower left corner of the dartboard that contains the point where the dart lands. That is, the lower left corner of B must be the same point as the lower left corner of the dartboard, and the dart lands somewhere along the upper or right edge of B . For X , find the CDF, PDF, $\mathbb{E}[X]$, and $\text{Var}(X)$.

Solution:

Since $F_X(x)$ is the probability that the dart lands inside the square of side length x , that probability is the area of a square of length x divided by the area of the square of length radius s (i.e., x^2/s^2). Thus, our CDF looks like

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2/s^2, & \text{if } 0 \leq x \leq s \\ 1, & \text{if } x > s \end{cases}$$

To find the PDF, we just need to take the derivative of the CDF, which gives us the following:

$$f_X(x) = \frac{d}{dx}F_X(x) = \begin{cases} 2x/s^2, & \text{if } 0 \leq x \leq s \\ 0, & \text{otherwise} \end{cases}$$

Using the definition of expectation and variance we can compute $\mathbb{E}[X]$ and $\text{Var}(X)$ in the following manner:

$$\mathbb{E}[X] = \int_0^s x f_X(x) dx = \int_0^s \frac{2x^2}{s^2} dx = \frac{2}{s^2} \int_0^s x^2 dx = \frac{2}{3s^2} [x^3]_0^s = \frac{2}{3}s$$

$$\mathbb{E}[X^2] = \int_0^s x^2 f_X(x) dx = \int_0^s \frac{2x^3}{s^2} dx = \frac{2}{s^2} \int_0^s x^3 dx = \frac{1}{2s^2} [x^4]_0^s = \frac{1}{2}s^2$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{1}{2}s^2 - \left(\frac{2}{3}s\right)^2 = \frac{1}{18}s^2$$