CSE 312 Section 3

Conditional Probability

Administrivia

Announcements & Reminders

- HW1
  - Grades released on Gradescope – check your submission to read comments
  - Regrade requests open ~24 hours after grades are released and close after a week
- HW2
  - Was due yesterday, Wednesday 7/5 @ 11pm
  - Late deadline Friday 7/7 @ 11pm (max of 2 late days per homework)
- HW3
  - Released on the course website
  - Due Wednesday 7/12 @ 11pm

Review

Any lingering questions from this last week?

Each week in section, we'll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we'll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Axioms of Probability and their Consequences

- **Axioms of Probability**
  - **Non-negativity:** For any event E, $\mathbb{P}(E) \geq 0$
  - **Normalization:** $\mathbb{P}(\Omega) = 1$
  - **Additivity:** If E and F are mutually exclusive events, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- **Corollaries of these axioms**
  - **Complementation**: $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
  - **Monotonicity**: If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  - **Inclusion-Exclusion**: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

- **Equally Likely Outcomes**: If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

Independence & Conditional Probability

- **Conditional Probability:** $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

- **Independent Events:**
  - Two events A, B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
  - If $\mathbb{P}(A) \neq 0$, this is equivalent to $\mathbb{P}(B|A) = \mathbb{P}(B)$.
  - If $\mathbb{P}(B) \neq 0$, this is equivalent to $P(A|B) = \mathbb{P}(A)$.

- **Bayes Theorem:** $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$

Partitioning the Space

- **Partition**: Nonempty events $E_1, \dots, E_n$ partition the sample space Ω iff
  - $E_1, \dots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
  - $E_1, \dots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
  - Note that for any event A (with $A \neq \emptyset,\ A \neq \Omega$): A and A^C partition Ω

- **Law of Total Probability (LTP):** Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$

Probability Definitions Continued

- **Bayes Theorem with LTP:** Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1)\mathbb{P}(A_1)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$.
  In particular, $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B|A)\mathbb{P}(A)+\mathbb{P}(B|A^C)\mathbb{P}(A^C)}$

- **Chain Rule:** Suppose $A_1, \dots, A_n$ are events. Then,
  $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2) \dots \mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1})$

Problem 3 – Game Show

3 – Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

3 – Game Show

2 rounds, contestants bribe in both rounds. If contestant has been bribing , allowed to stay with probability 1. If contestant has not been bribing, allowed to stay with probability 1/3, independent of earlier. Suppose that 1/4 of the contestants have been bribing the judges.

a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?
b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?
c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?
d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

Work on this problem with the people around you, and then we'll go over it together!

3 – Game Show

a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

3 – Game Show

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Let S_i be the event that she stayed during the i-th episode. By the Law of Total Probability conditioning on whether the contestant bribed the judges we get,

$$\mathbb{P}(S_1) = \mathbb{P}(\text{Bribe})\mathbb{P}(S_1|\text{Bribe}) + \mathbb{P}(\text{NoBribe})\mathbb{P}(S_1|\text{NoBribe})$$

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$$
\begin{aligned}
\mathbb{P}(S_1) = \mathbb{P}(\text{Bribe})\mathbb{P}(S_1|\text{Bribe}) + \mathbb{P}(\text{NoBribe})\mathbb{P}(S_1|\text{NoBribe}) \\
= \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} \\
= 1/2
\end{aligned}
$$

3 – Game Show

b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

3 – Game Show

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The event is $S_1 \cap S_2$. By the Law of Total Probability we get,

$$\mathbb{P}(S_1 \cap S_2) = \mathbb{P}(\text{Bribe})\mathbb{P}(S_1 \cap S_2|\text{Bribe}) + \mathbb{P}(\text{NoBribe})\mathbb{P}(S_1 \cap S_2|\text{NoBribe})$$

3 – Game Show

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If she hasn't been bribing judges, then the probability she stays on the show is 1/3, independent of what happens on earlier episodes. By conditional independence, we have:

$$= \frac{1}{4} \cdot 1 + \mathbb{P}(\text{NoBribe})\mathbb{P}(S_1|\text{NoBribe})\mathbb{P}(S_2|\text{NoBribe})$$

3 – Game Show

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If she hasn't been bribing judges, then the probability she stays on the show is 1/3, independent of what happens on earlier episodes. By conditional independence, we have:

$$= \frac{1}{4} \cdot 1 + \mathbb{P}(\text{NoBribe})\mathbb{P}(S_1|\text{NoBribe})\mathbb{P}(S_2|\text{NoBribe})$$
$$= \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3}$$
$$= \frac{1}{3}$$

3 – Game Show

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

3 – Game Show

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

By the definition of conditional probability and the Law of Total Probability,

$$\mathbb{P}(\overline{S_2}|S_1) = \frac{\mathbb{P}(S_1 \cap \overline{S_2})}{\mathbb{P}(S_1)} = \frac{\mathbb{P}(S_1 \cap \overline{S_2}|\text{Bribe})\mathbb{P}(\text{Bribe}) + \mathbb{P}(S_1 \cap \overline{S_2}|\text{NoBribe})\mathbb{P}(\text{NoBribe})}{\mathbb{P}(S_1)}$$

3 – Game Show

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

By the definition of conditional probability and the Law of Total Probability,

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If she has been bribing judges, then she is guaranteed to stay on the show, so

$$\mathbb{P}(S_1 \cap \overline{S_2}|\text{Bribe}) = \mathbb{P}(S_1|\text{Bribe})\mathbb{P}(\overline{S_2}|\text{Bribe}) = 1 \cdot 0 = 0$$

3 – Game Show

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

By the definition of conditional probability and the Law of Total Probability,

$$\mathbb{P}(\overline{S_2}|S_1) = \frac{\mathbb{P}(S_1 \cap \overline{S_2})}{\mathbb{P}(S_1)} = \frac{\mathbb{P}(S_1 \cap \overline{S_2}|\text{Bribe})\mathbb{P}(\text{Bribe}) + \mathbb{P}(S_1 \cap \overline{S_2}|\text{NoBribe})\mathbb{P}(\text{NoBribe})}{\mathbb{P}(S_1)}$$

If she has been bribing judges, then she is guaranteed to stay on the show, so

$$\mathbb{P}(S_1 \cap \overline{S_2}|\text{Bribe}) = \mathbb{P}(S_1|\text{Bribe})\mathbb{P}(\overline{S_2}|\text{Bribe}) = 1 \cdot 0 = 0$$

If she hasn't been bribing, then the probability she leaves is 2/3 by complementing, so

$$\mathbb{P}(S_1 \cap \overline{S_2}|\text{NoBribe}) = \mathbb{P}(S_1|\text{NoBribe})\mathbb{P}(\overline{S_2}|\text{NoBribe}) = \frac{1}{3} \cdot \frac{2}{3}$$

3 – Game Show

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

We already computed $\mathbb{P}(S_1)$ in part (a). Putting it all together we get:

$$\mathbb{P}(\overline{S_2}|S_1) = \frac{\mathbb{P}(S_1 \cap \overline{S_2}|\text{Bribe})\mathbb{P}(\text{Bribe}) + \mathbb{P}(S_1 \cap \overline{S_2}|\text{NoBribe})\mathbb{P}(\text{NoBribe})}{\mathbb{P}(S_1)}$$

$$= \frac{0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{3}{4}}{\frac{1}{2}}$$

$$= \frac{1/6}{1/2}$$

$$= \frac{1}{3}$$

3 – Game Show

d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

3 – Game Show

d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

Let B be the event that she bribed the judges. By Bayes' Theorem,

$$\mathbb{P}(B|S_1) = \frac{\mathbb{P}(S_1|B)\mathbb{P}(B)}{\mathbb{P}(S_1)} = \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Problem 4 – Parallel Systems

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

a) What is the probability the system is functioning?

b) If the system is functioning, what is the probability that component 1 is working?

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

Work on this problem with the people around you, and then we'll go over it together!

4 – Parallel Systems

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a) What is the probability the system is functioning?

Let C_i be the event component i is working, and F be the event that the system is functioning. For the system to function, it is sufficient for any component to be working. This means that the only case in which the system does not function is when none of the components work. We can then use complementing to compute $\mathbb{P}(F)$, knowing that $\mathbb{P}(C_i) = p$. We get:

$$\mathbb{P}(F) = 1 - \mathbb{P}(F^C) = 1 - \mathbb{P}\left(\bigcap_{i=1}^{n} C_i^C\right) = 1 - \prod_{i=1}^{n} \mathbb{P}(C_i^C)$$

$$= 1 - \prod_{i=1}^{n} \left(1 - \mathbb{P}(C_i)\right) = 1 - \prod_{i=1}^{n} (1 - p) = 1 - (1 - p)^n$$

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

b) If the system is functioning, what is the probability that component 1 is working?

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

b) If the system is functioning, what is the probability that component 1 is working?

We know that for the system to function only one component needs to be working, so for all i, we have $\mathbb{P}(F|C_i) = 1$. Using Bayes Theorem, we get

$$\mathbb{P}(C_1|F) = \frac{\mathbb{P}(F|C_1)\mathbb{P}(C_1)}{\mathbb{P}(F)} = \frac{1 \cdot p}{1-(1-p)^n} = \frac{p}{1-(1-p)^n}$$

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

$$\mathbb{P}(C_1|C_2, F) = \mathbb{P}(C_1|C_2) = \mathbb{P}(C_1) = p$$

The first equality holds because knowing C_2 and F is just as good as knowing C_2 (since if C_2 happens, F does too), and the second equality holds because the components working are independent of each other.

4 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

More formally, we can use the definition of conditional probability along with a careful application of the chain rule to get the same result. We start with the following expression:

$$\mathbb{P}(C_1|C_2,F) = \frac{\mathbb{P}(C_1,C_2,F)}{\mathbb{P}(C_2,F)} = \frac{\mathbb{P}(F|C_1,C_2)\mathbb{P}(C_1|C_2)\mathbb{P}(C_2)}{\mathbb{P}(F|C_2)\mathbb{P}(C_2)}$$

We note that the system is guaranteed to work if any one component is working, so $\mathbb{P}(F|C_1,C_2) = \mathbb{P}(F|C_2) = 1$. We also note that components work independently of each other, hence $\mathbb{P}(C_1|C_2) = \mathbb{P}(C_1)$. With that in mind, we can rewrite our expression so that:

$$\mathbb{P}(C_1|C_2,F) = \frac{1 \cdot \mathbb{P}(C_1)\mathbb{P}(C_2)}{1 \cdot \mathbb{P}(C_2)} = \mathbb{P}(C_1) = p$$

That's All, Folks!

Thanks for coming to section this week!
Any questions?