Section 2: Counting – Basic Discrete Probability

Review of Main Concepts (Counting)

- **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  In general: +singles - doubles + triples - quads + ...
- **Pigeonhole Principle:** If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.
- **Stars and bars:** The number of ways to distribute n indistinguishable balls into k distinguishable bins is $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$.
- **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events E and F is denoted $E \cup F$
- **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
- **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
- **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
- **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
- **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff
  - $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^n E_i = \Omega$, and
  - $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
    - ⋆ Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

Axioms of Probability and their Consequences

(a) **(Non-negativity)** For any event E, $\mathbb{P}(E) \geq 0$

(b) **(Normalization)** $\mathbb{P}(\Omega) = 1$

(c) **(Additivity)** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- **Corollaries of these axioms:**
  - $\mathbb{P}(E) + \mathbb{P}\left(E^C\right) = 1$
  - If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  - $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
- **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.
  - Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.
- **Conditional Probability**: $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

- **Independent Events.** Two events A, B are **independent** if $\mathbb{P}(A \cap B) =$ ____________.

  If $\mathbb{P}(A) \neq 0$, this is equivalent to $\mathbb{P}(B \mid A) =$ ____________.

  If $\mathbb{P}(B) \neq 0$, this is equivalent to $\mathbb{P}(A \mid B) =$ ____________.
- **Law of Total Probability (LTP)**: Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then

  $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(\mathrm{B} \mid A_i)\, \mathbb{P}(A_i)$

1. A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r} r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

2. GREED INNIT

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Repeat the question for the letters "AAAAABBB".

3. Friendships

Show that in any group n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

(a) Everyone has at least one friend.

(b) At least one person has no friends.

4. Powers and divisibility

Prove that there exist two powers of 7 whose difference is divisible by 2003. (You may want to use the Pigeonhole principle.)

5. Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?

6. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute $\mathbb{P}(E)$, where E is the event that the suits of the shuffled cards are in alternating order.

7. Balls from an Urn

Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

(a) Give a probability space describing the experiment.

(b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

(c) What is the probability that at most one ball is red?

(d) What is the probability that we get at least one green ball?

(e) Repeat **b)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn.

8. Weighted Die

Consider a weighted die such that

- $\mathbb{P}(1) = \mathbb{P}(2)$,
- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$, and
- $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$.

What is the probability that the outcome is 3 or 4?

9. Shuffling Cards

We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace $>$ King $>$ Queen $>$ Jack $>$ 10 $> \cdots >$ 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

10. Flipping Coins

A coin is tossed twice. The coin is “heads” one quarter of the time. You can assume that the second toss is independent of the first toss.

(a) What is the probability that the second toss is “heads” given that the first toss is “tails”?

(b) What is the probability that the second toss is “heads” given that at least one of the tosses is “tails”?

(c) In the probability space of this task, give an example of two events that are disjoint but not independent.

(d) In the probability space of this task, give an example of two events that are independent but not disjoint.

11. Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

12. Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

(a) What is the probability that all three balls are all of the same color?

(b) What is the probability that we get more than one red ball given the first ball is red?