# CSE 312 Section 2

## Intro to Discrete Probability

# Administrivia

# Announcements & Reminders

- Office Hours
  - Offered every day! Mostly in-person, a few zoom options
  - Times posted on the calendar on the course website

- HW1
  - Was due yesterday, Wednesday 6/28 @ 11pm
  - Late deadline Friday 6/30 @ 11pm (max of 2 late days per hw)

- HW2
  - Released on the course website
  - Due Wednesday 7/5 @ 11pm

# Homework

- Submissions
  - LaTeX (**recommended**)
    - overleaf.com
    - template and LaTeX guide posted on course website
- Homework will typically be due on Wednesdays at 11pm on Gradescope
- Each assignment can be submitted a max of 48 hours late
- You have **5 late days total** to use throughout the quarter
  - Anything beyond that will result in a deduction on further late assignments

# Review

# Any lingering questions from this last week?

Each week in section, we’ll be reviewing the main concepts from this week and putting them into action by going through some practice problems together.

But before we get into that review, we’ll try to start off each section with some time for you to ask questions.

- Was anything particularly confusing this week?
- Is there anything we can clarify before we dive into the review?

This is your chance to clear things up!

# More Counting

- **Binomial Theorem**: d $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- **Principle of Inclusion-Exclusion:**
  - 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
  - 3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
  - In general: + singles - doubles + triples - quads + …
- **Pigeonhole Principle:** If there are $n$ pigeons with $k$ holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$ ) pigeons
- **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

# Probability Definitions

- **Sample Space:** The set of all possible outcomes of an experiment, denoted $\Omega$ or $S$
- **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
- **Union:** The union of two events $E$ and $F$ is denoted $E \cup F$
- **Intersection:** The intersection of two events $E$ and $F$ is denoted $E \cap F$ or $EF$
- **Mutually Exclusive:** Events $E$ and $F$ are mutually exclusive iff $E \cap F = \emptyset$
- **Complement:** The complement of an event $E$ is denoted $E^C$ or $\bar{E}$ or $\neg E$, and is equal to $\Omega \backslash \mathrm{E}$
- **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$

# Probability Definitions cont.

- **Probability of an event $E$**: denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$

- **Partition**: Nonempty events $E_1, \ldots, E_n$ partition the sample space $\Omega$ iff
  - $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
  - $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
    - Note that for any event $A$ (with $A \neq \emptyset,\ A \neq \Omega$): $A$ and $A^C$ partition $\Omega$

# Axioms of Probability and their Consequences

- **Axioms of Probability**
  - **Non-negativity:** For any event $E$, $\mathbb{P}(E) \geq 0$
  - **Normalization:** $\mathbb{P}(\Omega) = 1$
  - **Additivity:** If $E$ and $F$ are mutually exclusive events, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

- **Corollaries of these axioms**
  - **Complementation**: $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
  - **Monotonicity**: If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  - **Inclusion-Exclusion**: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

- **Equally Likely Outcomes**: If every outcome in a finite sample space $\Omega$ is equally likely, and $E$ is an event, then $\mathbb{P}(E) = \frac{|E|}{|\Omega|}$

# Independence & Conditional Probability

- 
- **Conditional Probability:** $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

- **Independent Events:**
  - Two events $A, B$ are independent if $P(A \cap B) =$ ________.
  - If $P(A) \neq 0$, this is equivalent to $P(B|A) =$ ________.
  - If $P(B) \neq 0$, this is equivalent to $P(A|B) =$ ________.

- **Law of Total Probability (LTP):** Suppose $A_1, \ldots, A_n$ partition $\Omega$ and let $B$ be any event. Then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$

# Problem 1 - A Team and a Captain

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Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r$$

Hint: Consider two ways to choose a team of size $r$ out of a set of size $n$ and a captain of the team (who is also one of the team members).

Work on this problem with the people around you, and then we'll go over it together!

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Remember that a combinatorial proof just requires that we show both sides are equivalent ways of counting a situation.

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Left hand side:

1. first choosing the captain ($n$ choices)
2. and then choosing the rest of the team ($\binom{n-1}{r-1}$ choices).

Right hand side:

1. first choosing the team ($\binom{n}{r}$ choices)
2. and then choosing the captain from among the members of the team ($r$ choices).

# Problem 3 - Friendships

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Show that in any group $n$ people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if $A$ is friend of $B$, then $B$ is friend of $A$ – and nobody is a friend of themselves.) Solve in particular the following two cases individually:

a) Everyone has at least one friend.

b) At least one person has no friends.

Work on this problem with the people around you, and then we'll go over it together!

# 3 – Friendships

a) Everyone has at least one friend.

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Everyone has between 1 and $n - 1$ friends (i.e., $n - 1$ holes), and there are $n$ people (the "pigeons"). Therefore, two of them will have the same number of friends.

# 3 – Friendships

b) At least one person has no friends.

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Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are $n$ people (the "pigeons"). Therefore, two of them will have the same number of friends.

# Problem 5 - Count the Solutions

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Consider the following equation: $a_1 + a_2 + a_3 + a_4 + \ a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3,$ $a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some $a_i$.

How many different solutions are there to the equation?

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How many different solutions are there to the equation?

We should use stars and bars.

There are 70 ones (stars) and 6 blocks (6 – 1 = 5 bars). $\binom{70+6-1}{6-1} = \binom{75}{5}$

# Problem 7 – Balls from an Urn

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Say an urn contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls with replacement, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.
b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)
c) What is the probability that at most one ball is red?
d) What is the probability that we get at least one green ball?
e) Repeat b)-d) for the case where the balls are drawn without replacement, i.e., when the first ball is drawn, it is not placed back from the urn.

Work on this problem with the people around you, and then we'll go over it together!

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$$\mathbb{P}(\omega) = 1/9 \text{ for all } \omega \in \Omega$$

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The event is $B = \{RR\}$

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The probability of the event is $\mathbb{P}(B) = \frac{|B|}{9} = 1/9$

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The event is $C = \{BB, BR, BG, RB, RG, GB, GR, GG\}$

The probability of the event is $\mathbb{P}(C) = \frac{|C|}{9} = 8/9$

Alternatively, this is just $B^C$, the complement of $B$.

We know that $\mathbb{P}(B^C) = 1 - \mathbb{P}(B) = 1 - 1/9 = 8/9$

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The event is $D = \{BG, GB, RG, GR, GG\}$

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e) Repeat b)-d) for the case where **the balls are drawn without replacement.**

  a) Does the probability space change? If so, what is it now?

  b) What is the probability that both balls are red?

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$\Omega = \{BR, BG, RB, RG, GB, GR\}$

$\mathbb{P}(\omega) = \frac{1}{3} \cdot \frac{1}{2} = 1/6$ for all $\omega \in \Omega$ (3 choices for the first ball, 2 for the second)

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$B = \{\}$ We can't have both be red, so $\mathbb{P}(B) = 0$

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d) What is the probability that we get at least one green ball?

$D = \{BG, GB, RG, GR\}$ so $\mathbb{P}(D) = \frac{4}{6} = \frac{2}{3}$

# Problem 5 – Powers and Divisibility

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Prove that there exist two powers of 7 whose difference is divisible by 2003. (You may want to use the Pigeonhole principle.)

Work on this problem with the people around you, and then we'll go over it together!

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HINT: think about what are the pigeons, and what are the pigeonholes?

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There are 2003 possible remainders of division by 2003 (0 through 2002). Consider a sequence of powers $1, 7, 7^2, \dots, 7^{2003}$. It contains 2004 members, each of which can be expressed as $2003x_i + r_i$, where $x_i$ and $r_i$ are the quotient and remainder respectively for member $i$ for $0 \le i \le 2003$, when member $i$ is divided by 2003. Therefore, by Pigeonhole principle, there exists a pair of members, say $7^n$ and $7^m$, where $n > m$, such that $r_n = r_m$. Then their difference ($7^n - 7^m = 2003(x_n - x_m)$) is divisible by 2003.

It may be simpler to understand this with the help of a smaller example. Lets use 4 instead of 2003. The sequence would be therefore be 1,7,49,343,2401. The remainders after dividing by 4 would then be $1, 3, 1, 3, 1$. Lets take $7^1$ and $7^3$ as our $7^m$ and $7^n$ respectively. Then we find $7^3 - 7^1 = 343 - 7 = 336$. 336 is divisible by 4.

# Problem 10 – Weighted Die

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- Consider a weighted die such that
- $\mathbb{P}(1) = \mathbb{P}(2)$
- $\mathbb{P}(3) = \mathbb{P}(4) = \mathbb{P}(5) = \mathbb{P}(6)$
- $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$

What is the probability that the outcome is 3 or 4?

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What is the probability that the outcome is 3 or 4?

By the second axiom of probability, the sum of probabilities for the sample space must equal 1. That is, $\sum_{i=1}^{6} \mathbb{P}(i) = 1$. Since $\mathbb{P}(1) = \mathbb{P}(2)$ and $\mathbb{P}(1) = 3 \cdot \mathbb{P}(3)$, we have that:

$$\begin{aligned} 1 &= \mathbb{P}(1) + \mathbb{P}(2) + \mathbb{P}(3) + \mathbb{P}(4) + \mathbb{P}(5) + \mathbb{P}(6) \\ &= 3 \cdot \mathbb{P}(3) + 3 \cdot \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) + \mathbb{P}(3) = 10 \cdot \mathbb{P}(3) \end{aligned}$$

Thus, solving algebraically, $\mathbb{P}(3) = 0.1$, so $\mathbb{P}(3) = \mathbb{P}(4) = 0.1$. Since rolling a 3 and 4 are disjoint events, then

$\mathbb{P}(3 \text{ or } 4) = \mathbb{P}(3) + \mathbb{P}(4) = 0.1 + 0.1 = 0.2$

# Problem 2 - Subsubset

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Let $[n] = \{1,2, \dots, n\}$ denote the first $n$ natural numbers. How many (ordered) pairs of subsets $(A, B)$ are there such that $A \subseteq B \subseteq [n]$?

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Realize that, if there are no restrictions, for each element $i$ of $1, \dots, n$, there are four possibilities: it can be in only $A$, only $B$, both, or neither. In our case, there is only one that is not valid (violates $A \subseteq B$): being in $A$ but not $B$. Hence there are 3 choices for each element, so the total number of such ordered pairs of subsets is

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$$3^n$$

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Alternately, apply the sum rule by adding up the number of ways of doing this where $B$ has size $k$, where $k$ is any integer between 0 and $n$. Now apply the product rule to find the number of ways to choose $B$ of size exactly $k$ (there are $\binom{n}{k}$ possibilities for $B$), and then once $B$ is selected, count the number of ways of choosing $A$ which has to be a subset of $B$ ($2^k$ ways).

Hence, by the Binomial Theorem, the number of such ordered pairs of subsets is

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Hence, by the Binomial Theorem, the number of such ordered pairs of subsets is

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k} = 3^n$$

# Problem 3 – GREED INNIT

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Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent.

Repeat the question for the letters “AAAAABBB”.

Work on this problem with the people around you, and then we’ll go over it together!

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We use inclusion-exclusion. Let $\Omega$ be the set of all anagrams (permutations) of "INGREDIENT" and $A_I$ be the set of all anagrams with two consecutive I's. Define $A_E$ and $A_N$ similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. We use complementing to count the size of $\Omega \backslash (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ = singles – doubles + triples, and by complementing, $|\Omega \backslash (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of $A_I$ is the same as $A_E$ and $A_N$. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)$

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# 3 – GREED INNIT

Repeat the question for the letters “AAAAABBB”.

Note that no A’s and no B’s can be adjacent. So let us put the B’s down first:

_B_B_B_

By the pigeonhole principle, two A’s must go in the same slot, but then they would be adjacent, so there are no ways .

# Problem 9 – Shuffling Cards

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We have a deck of cards, with 4 suits, with 13 cards in each. Within each suit, the cards are ordered Ace > King > Queen > Jack > 10 > · · · > 2. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely). What is the probability the first card on the deck is (strictly) larger than the second one?

Work on this problem with the people around you, and then we'll go over it together!

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First off, the sample space $\Omega$ here consists of all pairs of cards – which we can represent by their value and suit, e.g., $(4\clubsuit, A\blacklozenge)$. There are $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}(\omega) = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event $E$ containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards $a$ and $b$ where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|E| = 16 \cdot 78 = 1248$$

Therefore, $\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47$

# That's All, Folks!

Thanks for coming to section this week!
Any questions?