CSE 312 Section 1

Combinatorics

Administrivia & Introductions

Announcements & Reminders

- Section Materials
  - Handouts will be provided in at each section
  - Worksheets and sample solutions will be available on the course calendar later this evening
- Office Hours
  - We start holding office hours today!
  - Times posted on the calendar on the course website
- HW1
  - Due Wednesday @ 11pm

Homework

- Submissions
  - LaTeX (highly encouraged)
    - overleaf.com
    - template and LaTeX guide posted on course website!
  - Word Editor that supports mathematical equations
  - Homework must be typeset!
- Homework will typically be due weekly at 11pm on Gradescope
- Each assignment can be submitted a max of 48 hours late
- You have **5 late days total** to use throughout the quarter
  - Only 1 late day can be used on the final homework
  - Anything beyond that will result in a deduction on further late assignments

Your TAs

- TA 1
  - content

- TA 2
  - content

Icebreaker

- Small groups
- Please share with your group
  - Your name
  - Number of years in department/ at UW
  - What plans do you have after the summer quarter?
  - What are you concerned about for 312 / what are you excited about?

Review

Any lingering questions from this last week?

Each week in section, we’ll be reviewing the main concepts from this week and putting them into action by going through some practice problems together. But before we get into that review, we’ll try to start off each section with some time for you to ask questions. Was anything particularly confusing this week? Is there anything we can clarify before we dive into the review? This is your chance to clear things up!

Some Counting Rules

- **Sum Rule:** If an experiment can either end up being one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is $N + M$.

- **Product Rule:** Suppose events $A_1, \dots, A_n$ each have $m_1, \dots, m_n$ possible outcomes, respectively. Then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i$ possible outcomes overall.

Counting the Number of Ways

- **Number of ways to order n distinct objects**: $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$
- **Number of ways to select from n distinct objects**:
  - **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n,k) = \frac{n!}{(n-k)!}$$

  - **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n,k)$$

Problem 2 - Seating

2 – Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if …

a) … all couples are to get adjacent seats?
b) … anyone can sit anywhere, except that one couple insists on not sitting in adjacent seats?

Work on this problem with the people around you, and then we'll go over it together!

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Consider each couple as a unit.

Apply the product rule, first choosing one of the 5! permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of 2^5). Therefore, the answer is:

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One way we can solve this is to consider by cases.

Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So, there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and 1 B can sit. Once they do, the other 8 people can sit in 8! ways since there are no other restrictions. Hence the total number of ways is

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$$(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!$$

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Another way to solve this is to apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats.

There are $10!$ for the former, and there are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple. That means the answer to the question is

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$$10! - 9! \cdot 2 = 8 \cdot 9!$$

Problem 6 – Birthday Cake

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A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Work on this problem with the people around you, and then we'll go over it together!

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Apply the product rule.

Start from Thursday and work forward and backward in the week: More precisely, given the 1 choice on Thursday, for each of Wednesday and Friday, there are 4 choices (the different pie options). Given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.

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Therefore the answer is

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = 46$$

Problem 9 – Photographs

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Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

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Apply the sum rule.

Count the number of ways the line can be organized if you are next to your friend. Then count the number of ways the line can be organized if there is one person between you and your friend. Then use the sum rule to add these up.

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Case 1: You are next to your friend. So we can think of you and your friend as being a "unit". Now apply the product rule: there are 7! ways to arrange the other 6 people together with the unit (of you and your friend). Once arranged, there are 2 ways to rearrange you and your friend in the order. So there are 7! * 2 ways to to line people up if you are next to your friend.

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Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Case 2: There is exactly 1 person between you and your friend. Apply the product rule by first picking the person who is between you (6 choices). Then, thinking of you, your friend and that person as a "unit", consider all arrangements of the 5 people plus the unit (6! ways). Finally, there are two ways for you and your friend to be placed within the trio. Therefore, altogether there are 6 * 6! * 2 possibilities.

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Finally we use the sum rule to add these two up:

$$(2 \cdot 7 + 2 \cdot 6)6!$$

Problem 5 – Escape the Professor

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There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Work on this problem with the people around you, and then we'll go over it together!

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Apply the product rule to first choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

The answer is

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Apply the product rule to first choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor (4 choices for the first, 3 for the second and so on).

The answer is

$$\binom{6}{4}\binom{7}{4}4!$$

That’s All, Folks!

Thanks for coming to section this week!
Any questions?