Joint Distributions

CSE 312 Su23
Lecture 16

Today

A closer look at more interesting probabilistic models!

Thus far, we’ve seen some basic models. A series of coin flips, car accidents, and the like.

Most interesting probabilistic models involve more than one random variable.

Modelling multiple RVs

Some common situations:

$\mathbb{P}(C \mid X_1 \cap X_2 \cap \dots)$

medical history, symptoms, personal information => predict illness type

traffic, pedestrians, weather, traffic lights => response of self-driving car

But wait!

We have looked at multiple random variables! We’ve looked at independence and such!

You’re right.

But what happens when we’re modelling multiple **dependent** random variables?

We’ll see this over the next few days…

Joint PMFs and Joint Range

For two (discrete) random variables X, Y, their **joint PMF** is:

$p_{X,Y}(a, b) = \mathbb{P}(X = a \cap Y = b) = \mathbb{P}(X = a, Y = b)$

$X: \{1, 2, 3\}$

$Y: \{1, 2\}$

The **joint range** of $p_{X,Y}$ is

$\Omega_{X,Y} = \{(c, d) : p_{X,Y}(c, d) > 0\} \subseteq \Omega_X \times \Omega_Y$

$X \times Y$ $(1,1)\,(1,2)$ $(2,1)\,(2,2)$ $(3,1)\,(3,2)$

Note: $\sum_x p_X(x) = 1$

$\star$ $\underbrace{\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s, t) = 1}$

$\sum_{s\in X} \sum_{t\in Y} p_{X,Y}(s,t) = 1$

$X = (3, 3)$
$Y =$

Examples

Roll a blue die and a red die. Each die is 4-sided. Let X be the blue die's result and Y be the red die's result.
Each die (individually) is fair. But not all results are equally likely when looking at them both together.

$p_{X,Y}(1,2) = 3/16.$

* Table of all possible outcomes for several discrete RVs.

Joint probability table

Ω_X

Ω_Y

$p_{X,Y}$	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	$\boxed{1/16}$	1/16	1/16	1/16
$Y=2$	3/16	0	0	1/16
$Y=3$	0	2/16	$\boxed{0}$	2/16
$Y=4$	0	$\boxed{1/16}$	3/16	0

$p_{X,Y}(1,1)$ $p_{X,Y}(3,3)$

$p_{X,Y}(2,4)$

Examples

$$\Omega_{X,Y} = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$$

$$\Omega_{X,Y} = \{(1,1), (1,2), (2,1), (2,3), (2,4), (3,1), (3,4), (4,1), (4,2), (4,3)\}$$

$p_{X,Y}$	X=1	X=2	X=3	X=4
Y=1	1/16	1/16	1/16	1/16
Y=2	3/16	0	0	1/16
Y=3	0	2/16	0	2/16
Y=4	0	1/16	3/16	0

$$\sum_{(s,t)\in\Omega_{X,Y}} p_{X,Y}(s,t) = 1$$

$$\frac{1+1+1+1+3+1+2+2+1+3}{16} = 1$$

Examples

$\mathbb{P}(X=1) = \mathbb{P}(X=2) = \mathbb{P}(X=3) = \mathbb{P}(X=4) = 1/4$

Roll a blue die and a red die. Each die is 4-sided. Let X be the blue die's result and Y be the red die's result.

Each die (individually) is fair. But not all results are equally likely when looking at them both together.

- mutually exclusive
- exhaustive

$p_{X,Y}$	$X=1$	$X=2$	$X=3$	$X=4$
$Y=1$	1/16	1/16	1/16	1/16
$Y=2$	3/16	0	0	1/16
$Y=3$	0	2/16	0	2/16
$Y=4$	0	1/16	3/16	0
	4/16	4/16	4/16	4/16

$\mathbb{P}(X=1) = \sum_{i=1}^{4} \mathbb{P}(X=1 \cap Y=i)$ LTP

Marginal Distributions

What if I just want to talk about X?

Well, use the law of total probability:

$\mathbb{P}(X = k) = \sum_{\text{partition } \{E_i\}} \mathbb{P}(X = k \cap E_i)$

and let E_i be possible outcomes for Y.

For the dice example

$\mathbb{P}(X = k) = \sum_{\ell=1}^{4} \mathbb{P}(X = k \cap Y = \ell)$

$p_X(k) = \sum_{\ell=1}^{4} p_{X,Y}(k, \ell)$

Marginal Distributions

$p_X(k) = \sum_{\ell=1}^{4} p_{X,Y}(k,\ell)$

So

$$p_X(2) = \frac{1}{16} + 0 + \frac{2}{16} + \frac{1}{16} = \frac{4}{16}$$

$\mathbb{P}(Y=1) = 4/16$

$\mathbb{P}(Y=2) = 4/16$

$\vdots$

$p_{X,Y}$	X=1	X=2	X=3	X=4
Y=1	1/16	1/16	1/16	1/16
Y=2	3/16	0	0	1/16
Y=3	0	2/16	0	2/16
Y=4	0	1/16	3/16	0

Marginal Distributions

In general, the marginal distributions of the joint PMF are defined as:

$$p_X(a) = \mathbb{P}(X = a) = \sum_y p_{X,Y}(a, y)$$

$$p_Y(b) = \mathbb{P}(Y = b) = \sum_x p_{X,Y}(x, b)$$

$p_X(k)$ is called the "marginal" distribution for X (because we "marginalized" Y) it's the same PMF we've always used; the name emphasizes we have gotten rid of one of the variables.

Weird dice

Roll two fair dice independently. Let X be the value of the first die, and Y be the value of the second die.

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and V be the maximum (i.e., $V = \max(X, Y)$)

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1				
V=2				
V=3				
V=4				

Weird dice

$\mathbb{P}(\underset{\text{min}}{U=1} \cap \underset{\text{max}}{V=1}) \rightarrow \frac{1}{16}$ $(1,1)$

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and V be the maximum (i.e., $V = \max(X, Y)$)

What is the joint distribution of U and V, $p_{U,V}(u, v)$?

$(1,2)$ $(2,1)$

$U = V$

$U < V$ $\mathbb{P}(\underset{\text{min}}{U=1} \cap \underset{\text{max}}{V=2}) = \frac{2}{16}$

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	$\frac{1}{16}$	0	0	0
V=2	$\frac{2}{16}$	$\frac{1}{16}$	0	0
V=3	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	0
V=4	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$U > V$

min > max

Weird dice

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and V be the maximum (i.e., $V = \max(X, Y)$)

What is the joint distribution of U and V, $p_{U,V}(u, v)$?

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

Weird dice

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and
V be the maximum (i.e., $V = \max(X, Y)$)

What is the joint distribution of U and V, $p_{U,V}(u, v)$?

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

$$p_{U,V}(u,v) = \begin{cases} 1/16 & (u,v) \in \Omega_U \times \Omega_V \text{ where } u = v \\ 2/16 & (u,v) \in \Omega_U \times \Omega_V \text{ where } v > u \\ 0 & \textit{otherwise} \end{cases}$$

$u > v$

Weird dice

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and

V be the maximum (i.e., $V = \max(X, Y)$)

What's $p_U(z)$? (the marginal for U)

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

$$p_U(z) = \mathbb{P}(U=z) = \begin{cases} 0 & \text{otherwise} \\ 7/16 & z=1 \\ 5/16 & z=2 \\ 3/16 & z=3 \\ 1/16 & z=4 \end{cases}$$

Weird dice

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and
V be the maximum (i.e., $V = \max(X, Y)$)

What's $p_U(z)$? (the marginal for U)

$$p_U(z) = \begin{cases} \frac{7}{16} & \text{if } z = 1 \\ \frac{5}{16} & \text{if } z = 2 \\ \frac{3}{16} & \text{if } z = 3 \\ \frac{1}{16} & \text{if } z = 4 \\ 0 & \text{otherwise} \end{cases}$$

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

Weird dice

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and

V be the maximum (i.e., $V = \max(X, Y)$)

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

Are U and V independent?

Recall: Discrete RVs U, V are independent iff $p_{U,V}(u, v) = p_U(u) \cdot p_V(v)$ for all $u \in \Omega_U, v \in \Omega_V$

Weird dice

Are U and V independent?

Recall: Discrete RVs U, V are independent iff $p_{U,V}(u, v) = p_U(u) \cdot p_V(v)$ for all $u \in \Omega_U, v \in \Omega_V$.

I *could* compute the marginal PMF of V, and then iterate through every pair of values of u and v, and check if these equations match.

But I am a bit lazy…

A check for independence

$P_{X,Y}(x,y) > 0$

The check: $\Omega_X \times \Omega_Y$ must equal $\Omega_{X,Y}$ for independence.

Suppose that there is some $(a, b) \in \Omega_X \times \Omega_Y$, but not in $\Omega_{X,Y}$.

$(a,b) \in \Omega_X \times \Omega_Y$

$a \in \Omega_X$, $b \in \Omega_Y$ $\Big\}$ $P_X(a) > 0$, $P_Y(b) > 0$.

Then:

$p_{U,V}(a, b) = 0$ 1

But:

$p_U(a) > 0$ and $p_U(b) > 0$

A check for independence

Suppose that there is some $(a, b) \in \Omega_X \times \Omega_Y$, but not in $\Omega_{X,Y}$.

Then: $p_{U,V}(a, b) = 0$ But: $p_U(a) > 0$ and $p_U(b) > 0$

Discrete RVs U, V are independent iff $\underline{p_{U,V}(u, v) = p_U(u) \cdot p_V(v)}$ for all $u \in \Omega_U, v \in \Omega_V$.

$= 0 \quad \neq \quad > 0 \quad > 0$

But the supposed event violates independence.

Therefore, a **necessary but not sufficient condition** for independence: $\Omega_X \times \Omega_Y = \Omega_{X,Y}$.

does not hold $\rightarrow$ X not idp Y

hold $\rightarrow$ check

(1,1) (1,2) (1,3) (1,4)

Weird dice

(4,4)

Let U be the minimum of the two rolls (i.e., $U = \min(X, Y)$) and

V be the maximum (i.e., $V = \max(X, Y)$)

$p_{U,V}$	U=1	U=2	U=3	U=4
V=1	1/16	0	0	0
V=2	2/16	1/16	0	0
V=3	2/16	2/16	1/16	0
V=4	2/16	2/16	2/16	1/16

Are U and V independent?

A **necessary but not sufficient condition** for independence: $\Omega_X \times \Omega_Y = \Omega_{X,Y}$.

U and V are <u>not independent</u>.

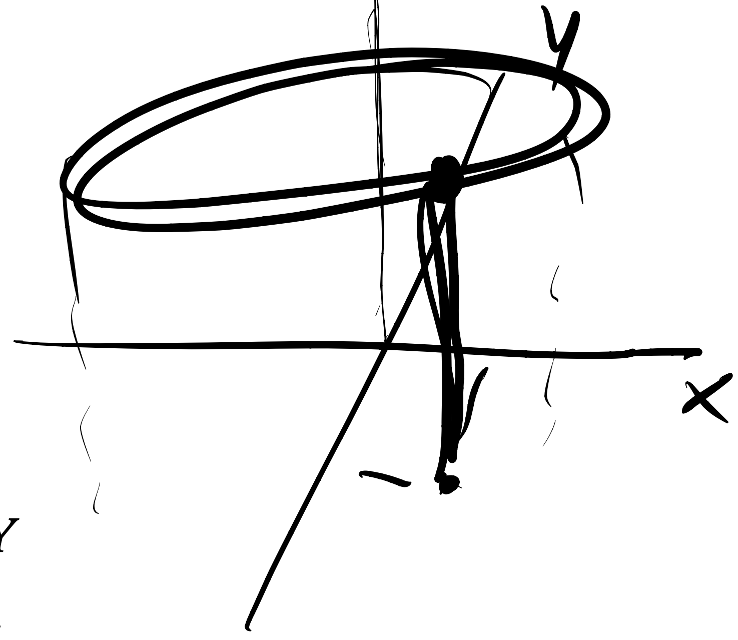
Joint PDFs…?

Yup.

A joint PDF for two continuous RVs X and Y is a function $f_{X,Y}(\mathrm{x},\mathrm{y})$ defined on $\mathbb{R}\times\mathbb{R}$ such that….

$f_{X,Y}(x,y) \geq 0\ for\ all\ x,y \in \mathbb{R}$ non-negative

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$ ✓



PDF

Joint range: $\Omega_{X,Y} = \{(c,d) : f_{X,Y}(c,d) > 0\} \subseteq \Omega_X \times \Omega_Y$

Joint continuous probabilities

$$\mathbb{P}(a \leq X \leq b \cap c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\, dy\, dx$$

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\, dx \quad \star$$

Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

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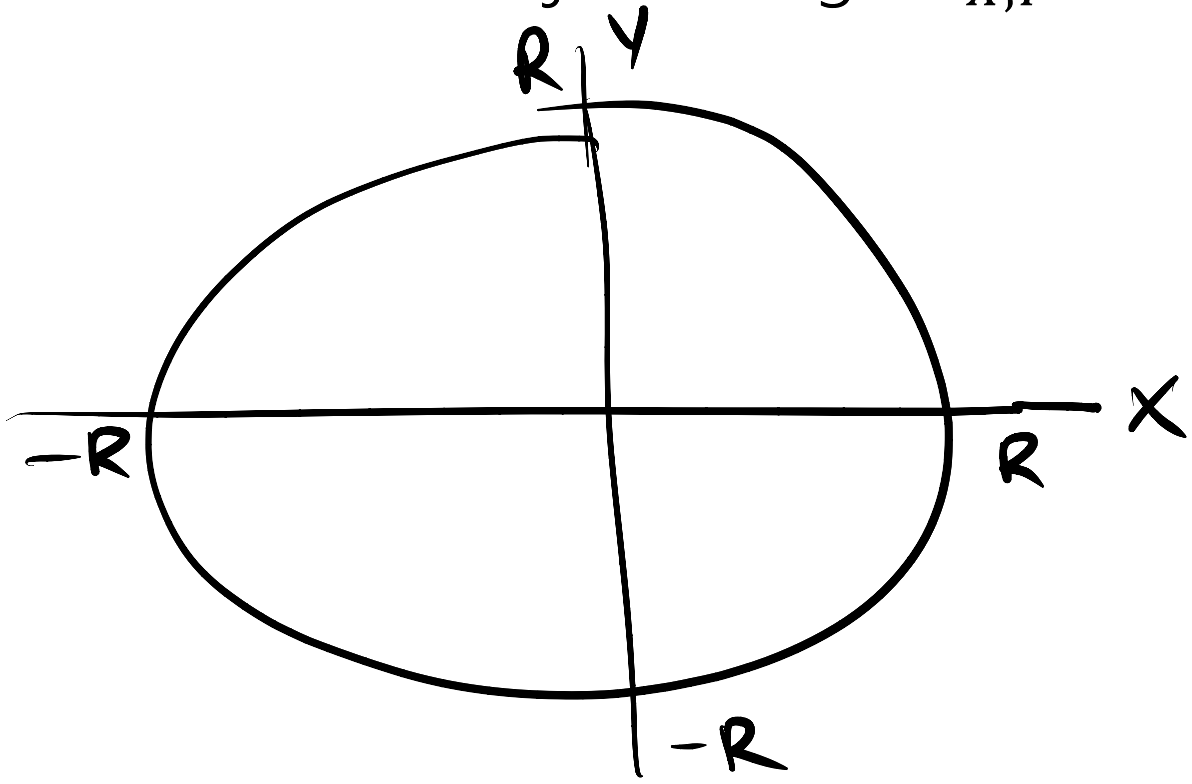
Let's find and sketch the joint range $\Omega_{X,Y}$.

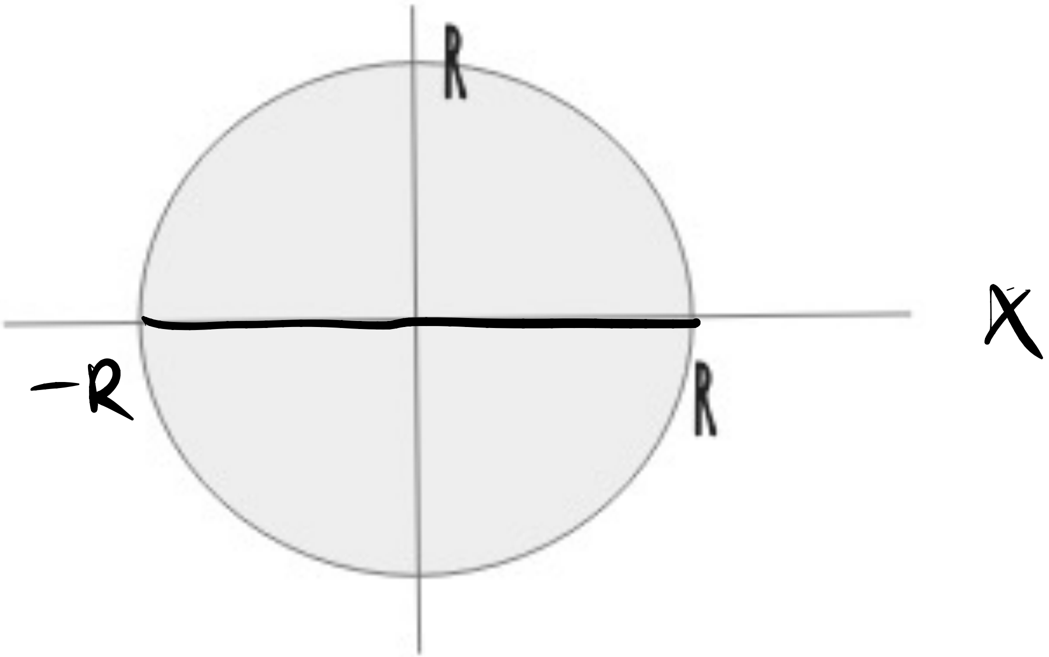


Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

Let's find and sketch the joint range $\Omega_{X,Y}$.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$



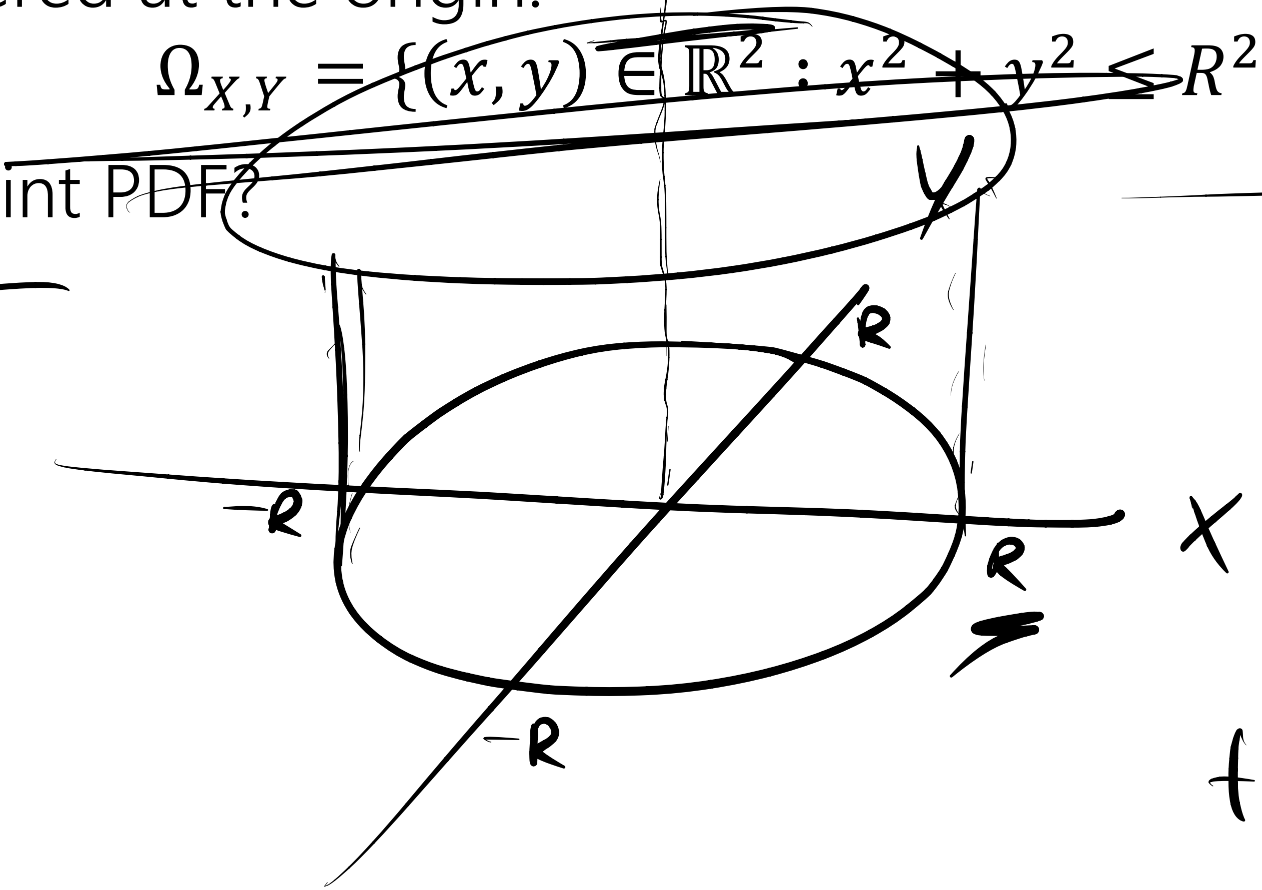
$x^2 + y^2 = r^2$

Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

What is the joint PDF?



Volume = 1

$\pi r^2 h = 1$

$\pi R^2 f_{X,Y}(x,y) = 1$

$$f_{X,Y}(x,y) = \frac{1}{\pi R^2}$$

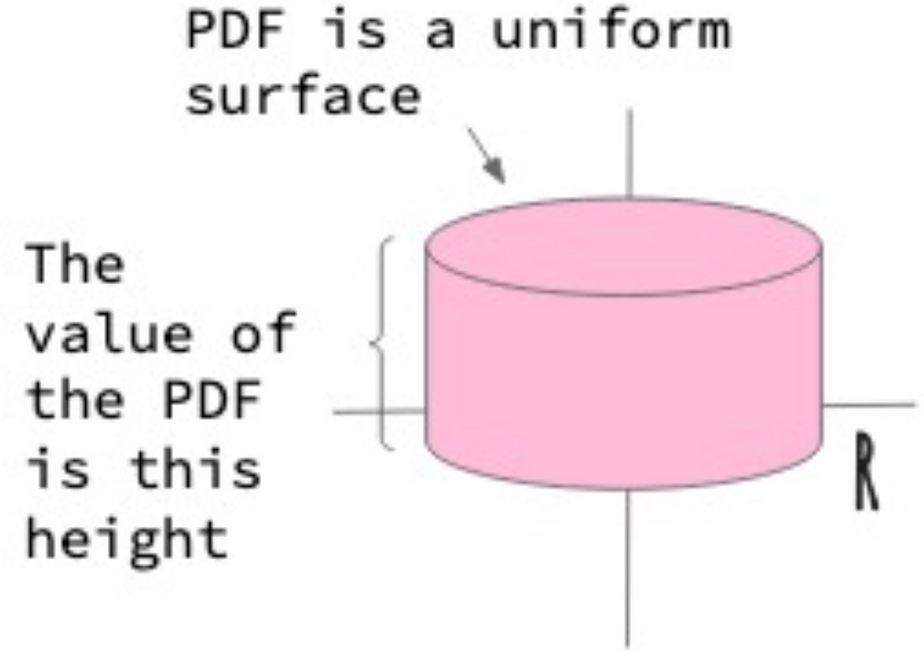
Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

What is the joint PDF?

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & otherwise \end{cases}$$



Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & otherwise \end{cases}$$

Ω_X: $[-R, R]$

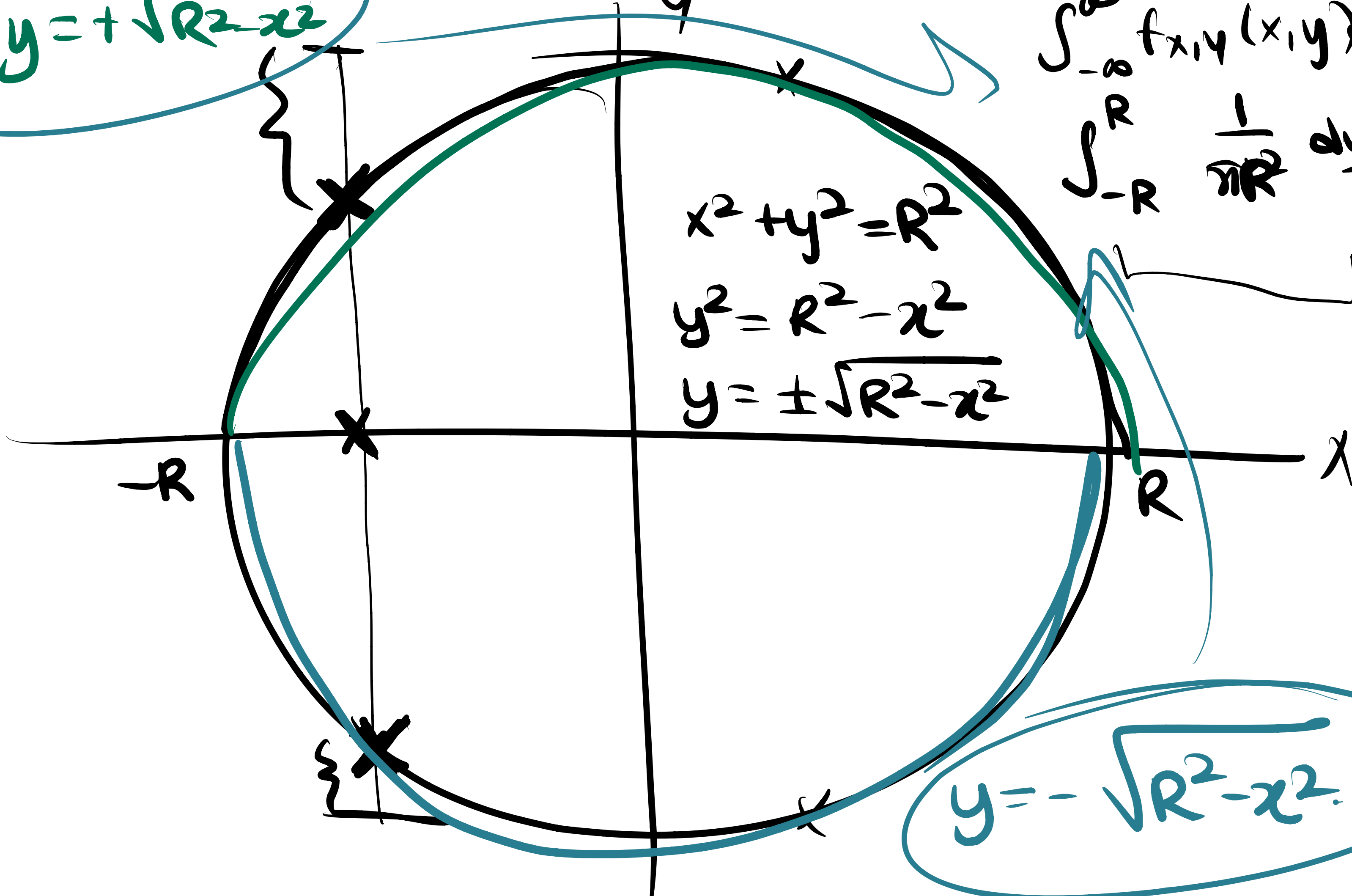
What about Ω_X and $f_X(x)$?

$$f_X(x) = \underbrace{\int_{-\infty}^{\infty}}_{} f_{X,Y}(x,y) \; dy$$

$$y = +\sqrt{R^2 - x^2}$$

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$\int_{-R}^{R} \frac{1}{\pi R^2}\,dy$$



$$x^2 + y^2 = R^2$$

$$y^2 = R^2 - x^2$$

$$y = \pm\sqrt{R^2 - x^2}$$

$$y = -\sqrt{R^2 - x^2}$$

Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & otherwise \end{cases}$$

What about Ω_X and $f_X(x)$?

$$\Omega_X = [-R, R]$$

$$f_X(x) = \int_{-\sqrt{R^2-x^2}}^{+\sqrt{R^2-x^2}} f_{X,Y}(x, y)\, dy$$

Darts

$\Omega_X \times \Omega_Y \neq \Omega_{X,Y}$

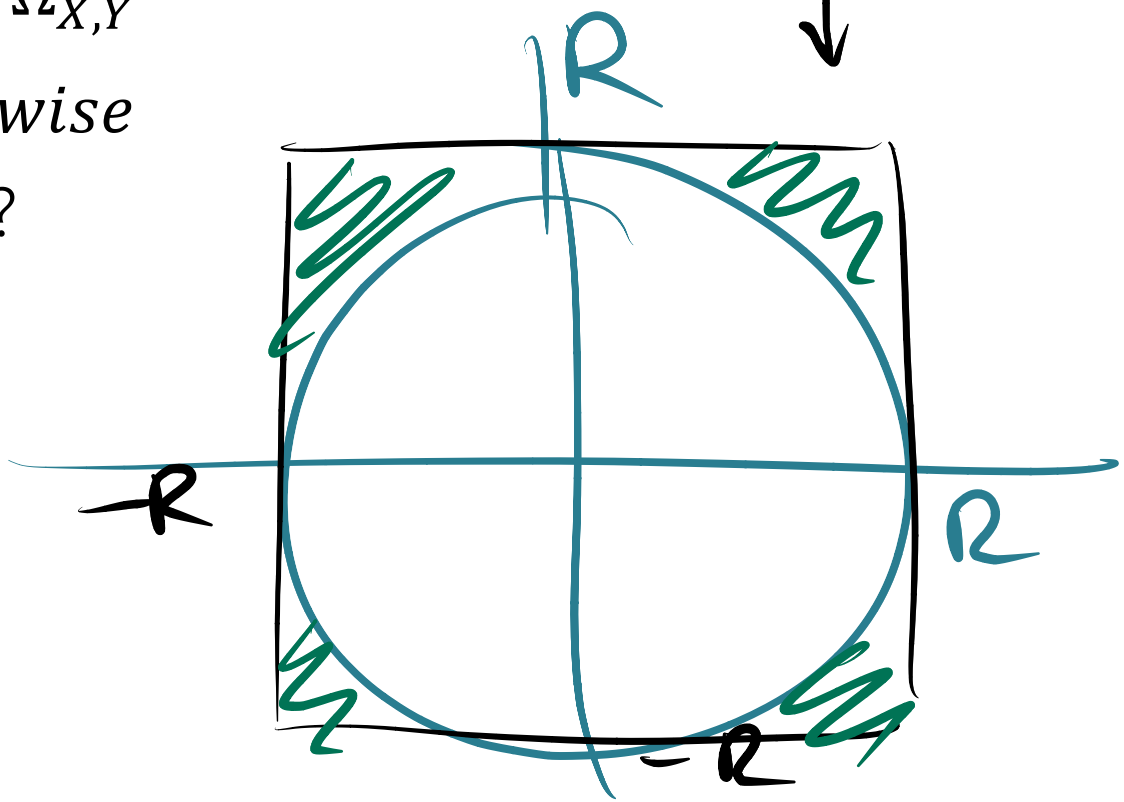
$[-R,R] \quad [-R,R]$

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & otherwise \end{cases}$$

Are X and Y independent?



Darts

Suppose X and Y are jointly and uniformly distributed on a circle of radius R centered at the origin.

$$\Omega_{X,Y} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}$$

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & x, y \in \Omega_{X,Y} \\ 0 & otherwise \end{cases}$$

Are X and Y independent? No, $\Omega_{X,Y} \neq \Omega_X \times \Omega_Y$.

Continuous Servers

The time until server 1 crashes is $X \sim Exp(\lambda)$ and the time until server 2 crashes (independent of server 1) is $Y \sim Exp(\mu)$.

What is the probability server 1 crashes before server 2?

$\mathbb{P}(X < Y)$

Continuous Servers

The time until server 1 crashes is $X \sim Exp(\lambda)$ and the time until server 2 crashes (independent of server 1) is $Y \sim Exp(\mu)$.

What is the probability server 1 crashes before server 2?

$\mathbb{P}(X < Y)$

$= \int_0^\infty \int_x^\infty f_{X,Y}(x, y)\, dy\, dx$

$= \int_0^\infty \int_x^\infty f_X(x) \cdot f_Y(y)\, dy\, dx$

Law of Total Probability (Continuous)

Let A be an event and Y a continuous random variable, then

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid Y = y) f_Y(y)\, dy$$

Joint Expectation

$\mathbb{P}(X=x \cap Y=y)$

Expectations of joint functions

For a function $g(X, Y)$, the expectation can be written in terms of the joint pmf.

$$\mathbb{E}[g(X,Y)] = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} g(x,y) \cdot p_{XY}(x,y)$$

This definition hopefully isn't surprising at this point (it's the value of g times the probability g takes on that value), but it's good to have down

$$\underline{\mathbb{E}[XY]} = \sum_{x \in \Omega_X} \sum_{y \in \Omega_Y} \underline{xy} \cdot \underline{p_{X,Y}(x,y)}$$

Conditional Distributions

A set of more formulae.

But really, explicitly just shifting our knowledge of conditional probabilities of events to random variables… to build up to an elegant result ☺

Conditional PMFs/PDFs

$\mathbb{P}(Y=b \mid X=a)$

$\cdot\, \mathbb{P}(X=a)$

$\mathbb{P}(X=a \cap Y=b)$

Conditional PMF:

$$p_{X|Y}(a \mid b) = \mathbb{P}(X = a \mid Y = b) = \frac{p_{X,Y}(a,b)}{p_Y(b)} = \frac{p_{Y|X}(b \mid a)\, p_X(a)}{p_{Y(b)}}$$

Cond prod Bayes

Conditional PDF:

$$f_{X|Y}(a \mid b) = \frac{f_{X,Y}(a,b)}{f_Y(b)} = \frac{f_{Y|X}(b \mid a)\, f_X(a)}{f_{Y(b)}}$$

Cond

Conditional Expectation

Waaaaaay back when, we said conditioning on an event creates a new probability space, with all the laws holding.

So, we can define things like "conditional expectations" which is the expectation of a random variable in that new probability space.

RV event

$$\mathbb{E}[X|A] = \sum_{x\in\Omega} x \cdot \mathbb{P}(X = x|A)$$

$$\mathbb{E}[X|Y = y] = \sum_{x\in\Omega_X} x \cdot \mathbb{P}(X = x|Y = y)$$

Conditional Expectations

All your favorite theorems are still true.

For example, linearity of expectation still holds

$$\mathbb{E}[(aX + bY + c)\ |A] = a\mathbb{E}[X|A] + b\mathbb{E}[Y|A] + c$$

Law of Total Expectation

Let $A_1, A_2, \dots, A_k$ be a partition of the sample space, then

$$\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{E}[X|A_i]\mathbb{P}(A_i)$$

$$\mathbb{P}(B) = \sum_{i=1}^{k} \mathbb{P}(B|E_i) \cdot \mathbb{P}(E_i) \quad \text{LTP}$$

$E_1 \cdots E_k$

Similar in form to law of total probability, and the proof goes that way as well.

Law of Total Expectation

Let $A_1, A_2, \ldots, A_k$ be a partition of the sample space, then

$$\mathbb{E}[X] = \sum_{i=1}^{k} \mathbb{E}[X|A_i]\mathbb{P}(A_i)$$

Let X, Y be discrete random variables, then

$$\mathbb{E}[X] = \sum_{y \in \Omega_Y} \mathbb{E}[X|Y = y]\mathbb{P}(Y = y)$$

Similar in form to law of total probability, and the proof goes that way as well.

LTE

$Y \mid X=0 \sim \underbrace{\mathrm{Geo}(1)}$ $\qquad \Omega_X : \{\underbrace{0,1,2}\}$

You will flip 2 (independent, fair coins). Call the number of heads X. Then (independently of the coin flips) draw a geometric random variable Y from the distribution $\mathrm{Geo}(X+1)$.

What is $\mathbb{E}[Y]$?

$$\mathbb{E}[Y] = \sum_{i=0}^{2} \mathbb{E}[Y \mid X=i] \cdot \mathbb{P}(X=i) \qquad \text{LTE}$$

$$= \underbrace{\mathbb{E}[Y \mid X=0)}\,\mathbb{P}(X=0) + \mathbb{E}[Y \mid X=1]\,\mathbb{P}(X=1) + \mathbb{E}[Y \mid X=2)\,\mathbb{P}(X=2)$$

$$= 1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4}$$

LTE

You will flip 2 (independent, fair coins). Call the number of heads X. Then (independently of the coin flips) draw a geometric random variable Y from the distribution $Geo(X+1)$.

What is $\mathbb{E}[Y]$?

$\mathbb{E}[Y]$

$= \mathbb{E}[Y|X=0]\mathbb{P}(X=0) + \mathbb{E}[Y|X=1]\mathbb{P}(X=1) + \mathbb{E}[Y|X=2]\mathbb{P}(X=2)$

$= \mathbb{E}[Y|X=0] \cdot \frac{1}{4} + \mathbb{E}[Y|X=1] \cdot \frac{1}{2} + \mathbb{E}[Y|X=2] \cdot \frac{1}{4}$

$= \frac{1}{0+1} \cdot \frac{1}{4} + \frac{1}{1+1} \cdot \frac{1}{2} + \frac{1}{2+1} \cdot \frac{1}{4} = \frac{7}{12}$.

Elevator Rides

X people enter an elevator on the ground floor, where $X \sim Poi(10)$.

There are N floors above the ground floor, and each person is equally likely to get off at any one of the N floors (independent of where others get off).

What is the expected number of stops the elevator makes before all passengers get off?

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Analogues for continuous

Everything we saw today has a continuous version.

There are "no surprises"– replace pmf with pdf and sums with integrals.

	Discrete	**Continuous**
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X=x, Y=y)$	$f_{X,Y}(x,y) \neq P(X=x, Y=y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\,ds\,dt$
Normalization	$\sum_x \sum_y p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$
Expectation	$E[g(X,Y)] = \sum_x \sum_y g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$
Conditional PMF/PDF	$p_{X\mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y=y] = \sum_x x p_{X\mid Y}(x \mid y)$	$E[X \mid Y=y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x \mid y)\,dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$