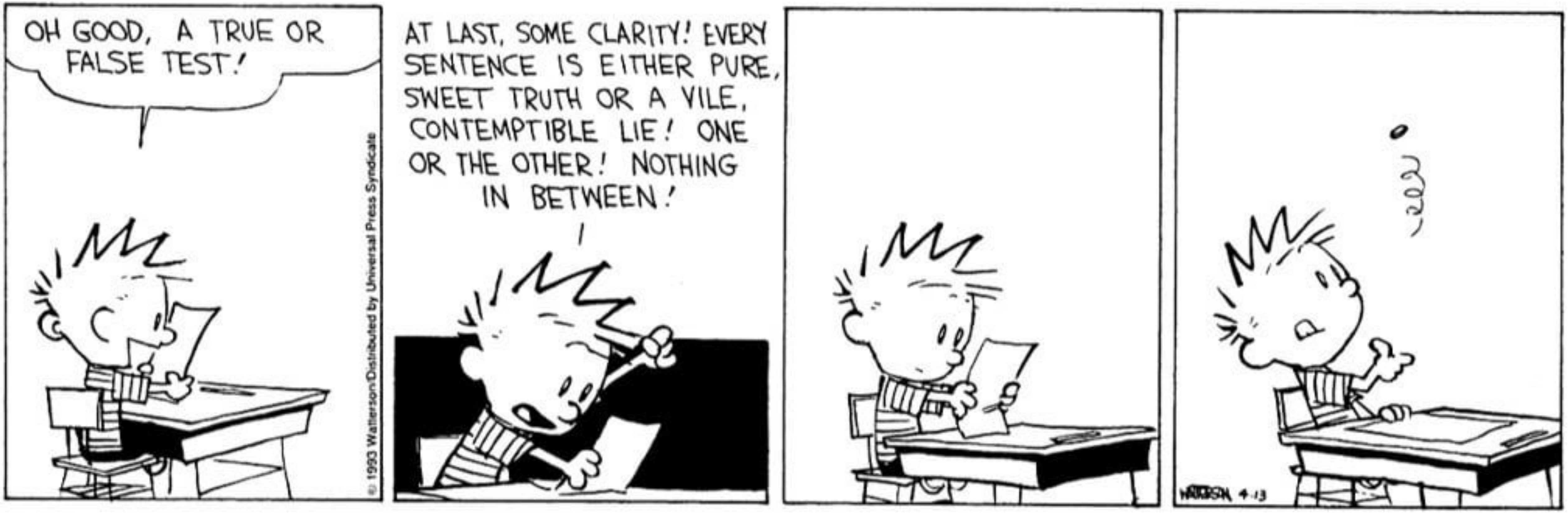



# Discrete RV Zoo: Part I

CSE 312 Su23
Lecture 10

# Announcements

Midterm review session – 1.10-2.10pm on Friday in PCAR 391

HW3 due today

# What Does Independence Give Us?

If $X$ and $Y$ are independent random variables, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$$

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\mathbb{E}[Y]$$

# Shifting a random variable

For any random variable $\mathbf{X}$, and any constants $\mathbf{a}, \mathbf{b}$:

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

For any random variable $\mathbf{X}$, and any constants $\mathbf{a}, \mathbf{b}$:

$$\mathbf{Var}(aX + b) = a^2\mathbf{Var}(\mathbf{X})$$

# Discrete Random Variable Zoo

There are common **patterns** of experiments:

Flip a [fair/unfair] coin [blah] times and count the number of heads.

Flip a [fair/unfair] coin until the first time that you see a heads

Draw a uniformly random element from [set]

Define an indicator random variable for [event]

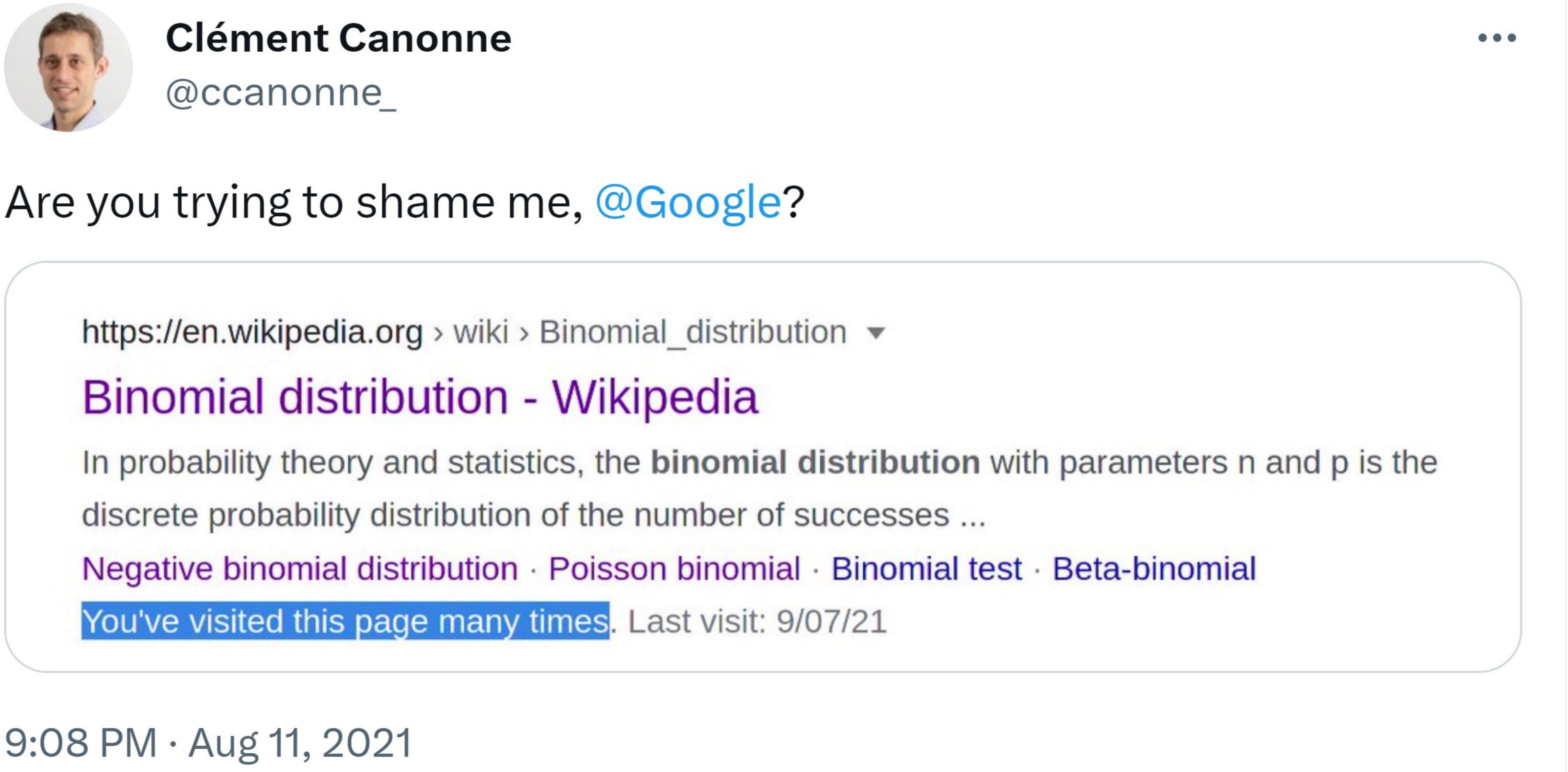
…

Instead of calculating the pmf, cdf, support, expectation, variance,… every time, why not calculate it **once** and look it up every time?

# What’s our goal?

Your goal is NOT to memorize these facts (it’ll be convenient to memorize some of them, but don’t waste time making flash cards).

Everything is on Wikipedia anyway. Everyone checks Wikipedia when they forget these.



# What’s our goal?

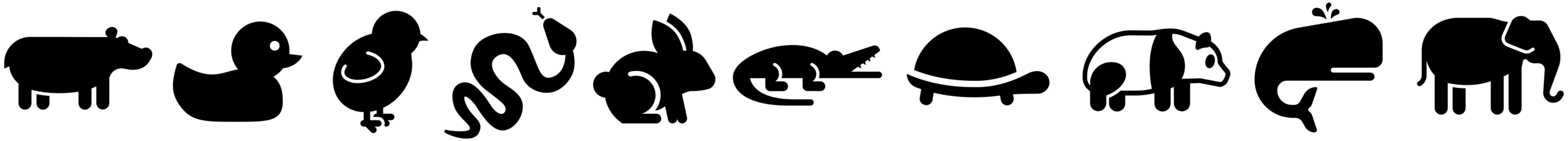
Your goal is NOT to memorize these facts (it’ll be convenient to memorize some of them, but don’t waste time making flash cards).

Everything is on Wikipedia anyway. Everyone checks Wikipedia when they forget these.

Our goal is to learn to recognize the characteristics of common distributions to identify which is relevant to our problem.

(And apply what we’ve learnt about probabilities, expectations and variances)

# Zoo!



### $X \sim \mathrm{Unif}(a, b)$

$$p_X(k) = \frac{1}{b - a + 1}$$
$$\mathbb{E}[X] = \frac{a + b}{2}$$
$$\mathrm{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

### $X \sim \mathrm{Ber}(p)$

$$p_X(0) = 1 - p;$$
$$p_X(1) = p$$
$$\mathbb{E}[X] = p$$
$$\mathrm{Var}(X) = p(1 - p)$$

### $X \sim \mathrm{Bin}(n, p)$

$$p_X(k) = \binom{n}{k} p^k (1 - p)^{n-k}$$
$$\mathbb{E}[X] = np$$
$$\mathrm{Var}(X) = np(1 - p)$$

### $X \sim \mathrm{Geo}(p)$

$$p_X(k) = (1 - p)^{k-1} p$$
$$\mathbb{E}[X] = \frac{1}{p}$$
$$\mathrm{Var}(X) = \frac{1 - p}{p^2}$$

### $X \sim \mathrm{NegBin}(r, p)$

$$p_X(k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$
$$\mathbb{E}[X] = \frac{r}{p}$$
$$\mathrm{Var}(X) = \frac{r(1 - p)}{p^2}$$

### $X \sim \mathrm{HypGeo}(N, K, n)$

$$p_X(k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$
$$\mathbb{E}[X] = n\frac{K}{N}$$
$$\mathrm{Var}(X) = \frac{K(N - K)(N - n)}{N^2(N - 1)}$$

### $X \sim \mathrm{Poi}(\lambda)$

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$
$$\mathbb{E}[X] = \lambda$$
$$\mathrm{Var}(X) = \lambda$$

For each RV in our zoo, we will look at:

> The type of situation being modelled

> Name & parameters

> The PMF, expectation and variance

> Example scenarios for use

# Some More Familiar Variables

# Situation: Bernoulli

Takes on two values: 1 or 0 (like a Boolean!)

# Situation: Bernoulli

Takes on two values: 1 or 0 (like a Boolean!)

Familiar examples…

You flip a coin (once) and want to record whether the flip was Heads.

You define an indicator random variable and want to know whether it's 1 or not.

More generally: you have one trial, and some probability $p$ of "success."

# Bernoulli Distribution

$X \sim \mathrm{Ber}(p)$

Parameter $p$ is probability of success (value 1)

# Bernoulli Distribution

$X \sim \text{Ber}(p)$

Parameter $p$ is probability of success (value 1)

$p_X(0) = 1 - p, p_X(1) = p$

$$F_X(k) = \begin{cases} 0 & \text{if } k < 0 \\ 1 - p & \text{if } 0 \leq k < 1 \\ 1 & \text{if } k \geq 1 \end{cases}$$

$\mathbb{E}[X] = p$

$\text{Var}(X) = p(1 - p)$

# Applications: Bernoulli Distribution

Did a particular bit get written correctly on the device?

Did you guess right on a multiple-choice test?

Did a server in a cluster fail?

Did a disk drive crash?

Did buying a particular share of a stock pay off?

Did a user like or dislike a YouTube video? (Back in ye olden days, at least)

Does a user click an ad?

# Situation: Binomial

You flip a coin $n$ times independently, each with a probability $p$ of coming up heads. How many heads are there?

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You flip a coin $n$ times independently, each with a probability $p$ of coming up heads. How many heads are there?

More generally: How many success did you see in $n$ independent (Bernoulli!) trials, where each trial has probability $p$ of success?

# Binomial Distribution

$X \sim \mathrm{Bin}(n, p)$ (X: number of successes across the $n$ trials)

$n$ is the number of independent trials

$p$ is the probability of success for one trial

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$X \sim \text{Bin}(n, p)$ (X: number of successes across the $n$ trials)

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$p$ is the probability of success for one trial

$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, 1, \dots, n\}$

$F_X$ is ugly.

$\mathbb{E}[X] = np$

$\text{Var}(X) = np(1-p)$

# Applications: Binomial Distribution

How many bits were written correctly on the device?

How many questions did you guess right on a multiple-choice test?

How many servers in a cluster failed?

How many keys went to one bucket in a hash table?

# Changing the world via ads…

A website serves 100 ads to a user in a day. The probability that the user clicks on an ad is 0.01. What is the probability of 10 clicks?

What is the probability of at least 3 clicks?

# Situation: Geometric

You flip a coin (which comes up heads with probability $p$) until you get a heads. How many flips did you need?

More generally: how many independent trials are needed until (up to and including) the first success?

# Geometric Distribution

$X \sim \mathrm{Geo}(p)$

$p$ is the probability of success for one trial

$X$ is the number of independent trials until the first success

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$X \sim \mathrm{Geo}(p)$

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$p_X(k) = (1-p)^{k-1}p$ for $k \in \{1,2,3,\dots\}$

$F_X(k) = 1-(1-p)^k$ for $k \in \mathbb{N}$

$\mathbb{E}[X] = \frac{1}{\mathrm{p}}$

$\mathrm{Var}(X) = \frac{1-p}{p^2}$

# Applications: Geometric Distribution

How many bits can we write before one is incorrect?

How many questions do you have to answer until you get one right?

How many times can you run an experiment until it fails for the first time?

# Geometric: Analysis

Both the expectation and variance are new to us.

The derivations of both are uninformative

Every derivation I’ve ever seen has wild algebra tricks.

# Geometric: Expectation

$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$

$= p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$

# Geometric: Expectation

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

# Geometric: Expectation

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p$$

$$= p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Intuition: Smaller $p$ means longer wait

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Intuition: for small $p$ lots of variance (might have to wait a long time, and it's variable)
For large $p$ very little variance (for $p = 1$ there's no variation at all!)

# Geometric Property

Geometric random variables are called “memoryless”

Suppose you’re flipping coins (independently) until you see a heads.

The first three came up tails.

How many flips are ***left*** until you see the first heads?

It’s another independent copy of the original!

The coin “forgot” it already came up tails 3 times.

# Formally…

Let $X$ be the number of flips needed until 1st H, $Y$ be the flips after TTT until first H.

$\mathbb{P}(Y = k | X \geq 3) =?$

…

Which is $p_X(k)$.

# Formally…

Let $X$ be the number of flips needed, $Y$ be the flips after the third.

$$\mathbb{P}(Y = k | X \geq 3) = \frac{\mathbb{P}(Y=k \cap X \geq 3)}{\mathbb{P}(X \geq 3)}$$

$$= \frac{(1-p)^{k-1+3}p}{(1-p)^3}$$

$$= (1-p)^{k-1}p$$

Which is $p_X(k)$.

# Another formulation of memorylessness

$$\mathbb{P}(X > n + m \mid X > m)$$

$$= \frac{\mathbb{P}(X>n+m \cap X>m)}{\mathbb{P}(X>m)}$$

$$= \frac{\mathbb{P}(X>n+m)}{\mathbb{P}(X>m)} = \frac{(1-p)^{n+m}}{(1-p)^m}$$

$$= (1-p)^n$$

$$= \mathbb{P}(X > n)$$

**Memorylessness**

A random variable X is memoryless if for all $s, t \geq 0$

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t)$$

# Scenario: Uniform

You roll a fair die (or draw a random integer from 1,…,n).

More generally: you want an integer in some range, with each equally likely.

# Discrete Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$

Parameter $a$ is the minimum value in the support, $b$ is the maximum value in the support.

$X$ is a uniformly random integer between $a$ and $b$ (inclusive)

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Parameter $a$ is the minimum value in the support, $b$ is the maximum value in the support.

$X$ is a uniformly random integer between $a$ and $b$ (inclusive)

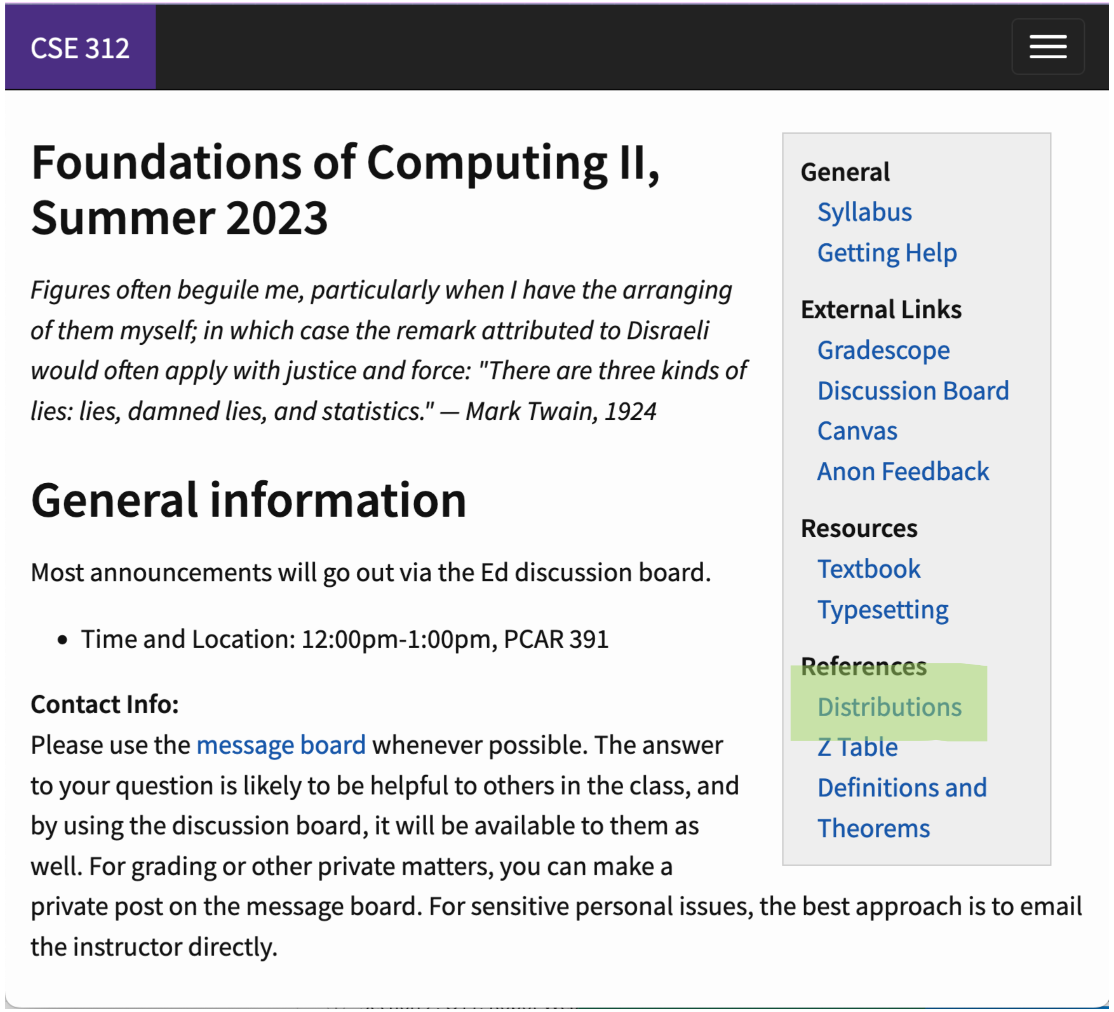
$p_X(k) = \frac{1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$

$F_X(k) = \frac{k-a+1}{b-a+1}$ for $k \in \mathbb{Z}$, $a \leq k \leq b$.

$\mathbb{E}[X] = \frac{a+b}{2}$

$\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

# $FREE.99 distributions cheat sheet >:)



CSE 312

## Foundations of Computing II, Summer 2023

*Figures often beguile me, particularly when I have the arranging of them myself; in which case the remark attributed to Disraeli would often apply with justice and force: "There are three kinds of lies: lies, damned lies, and statistics." — Mark Twain, 1924*

## General information

Most announcements will go out via the Ed discussion board.

- Time and Location: 12:00pm-1:00pm, PCAR 391

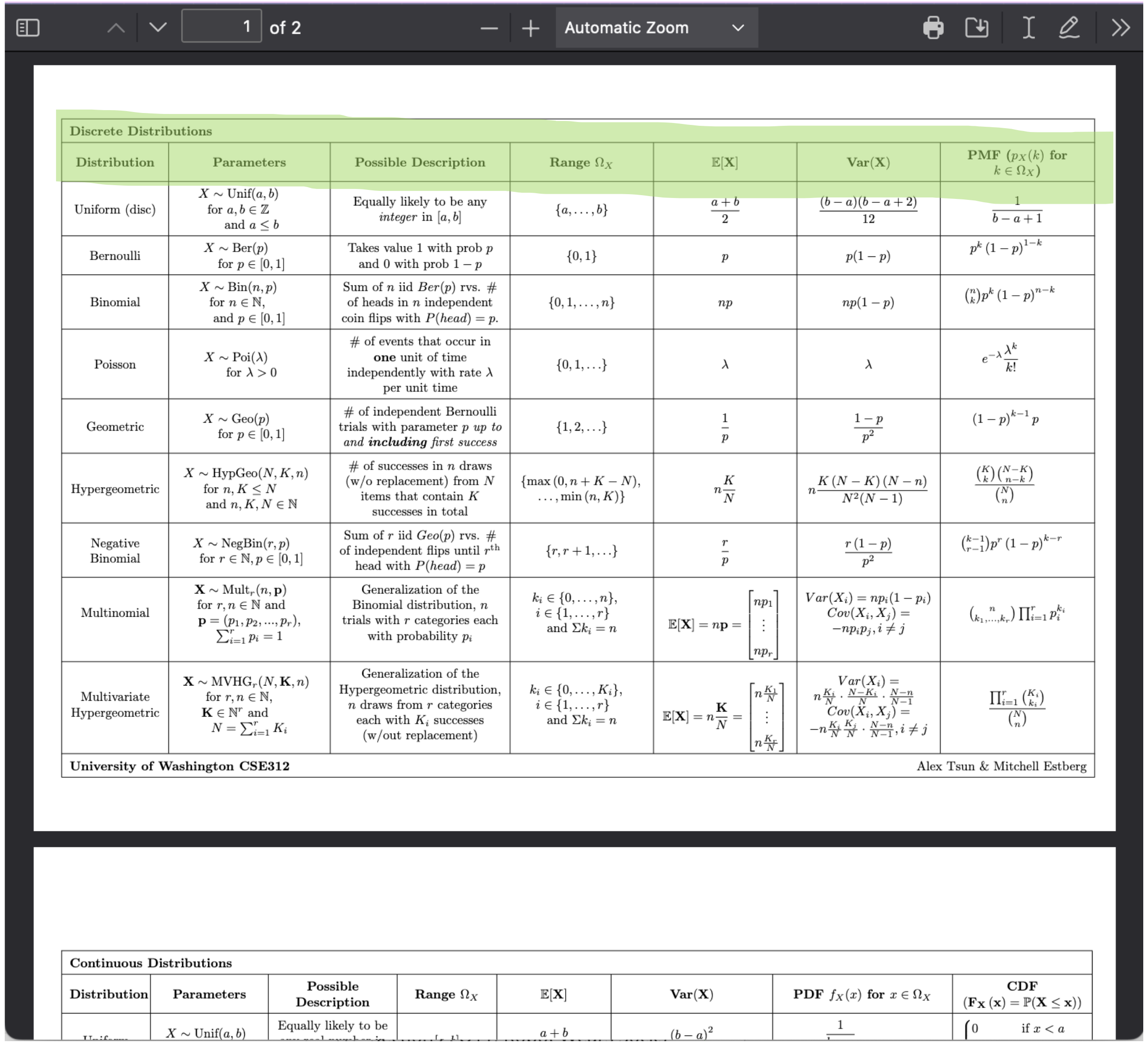
**Contact Info:**
Please use the message board whenever possible. The answer to your question is likely to be helpful to others in the class, and by using the discussion board, it will be available to them as well. For grading or other private matters, you can make a private post on the message board. For sensitive personal issues, the best approach is to email the instructor directly.

**General**
- Syllabus
- Getting Help

**External Links**
- Gradescope
- Discussion Board
- Canvas
- Anon Feedback

**Resources**
- Textbook
- Typesetting

**References**
- Distributions
- Z Table
- Definitions and Theorems





**Discrete Distributions**

| Distribution | Parameters | Possible Description | Range $\Omega_X$ | $\mathbb{E}[X]$ | Var(X) | PMF ($p_X(k)$ for $k \in \Omega_X$) |
|---|---|---|---|---|---|---|
| Uniform (disc) | $X \sim \text{Unif}(a,b)$ for $a,b \in \mathbb{Z}$ and $a \le b$ | Equally likely to be any *integer* in $[a,b]$ | $\{a,\dots,b\}$ | $\frac{a+b}{2}$ | $\frac{(b-a)(b-a+2)}{12}$ | $\frac{1}{b-a+1}$ |
| Bernoulli | $X \sim \text{Ber}(p)$ for $p \in [0,1]$ | Takes value 1 with prob $p$ and 0 with prob $1-p$ | $\{0,1\}$ | $p$ | $p(1-p)$ | $p^k(1-p)^{1-k}$ |
| Binomial | $X \sim \text{Bin}(n,p)$ for $n \in \mathbb{N}$, and $p \in [0,1]$ | Sum of $n$ iid $Ber(p)$ rvs. # of heads in $n$ independent coin flips with $P(head)=p$. | $\{0,1,\dots,n\}$ | $np$ | $np(1-p)$ | $\binom{n}{k}p^k(1-p)^{n-k}$ |
| Poisson | $X \sim \text{Poi}(\lambda)$ for $\lambda > 0$ | # of events that occur in **one** unit of time independently with rate $\lambda$ per unit time | $\{0,1,\dots\}$ | $\lambda$ | $\lambda$ | $e^{-\lambda}\frac{\lambda^k}{k!}$ |
| Geometric | $X \sim \text{Geo}(p)$ for $p \in [0,1]$ | # of independent Bernoulli trials with parameter $p$ *up to and **including** first success* | $\{1,2,\dots\}$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ | $(1-p)^{k-1}p$ |
| Hypergeometric | $X \sim \text{HypGeo}(N,K,n)$ for $n,K \le N$ and $n,K,N \in \mathbb{N}$ | # of successes in $n$ draws (w/o replacement) from $N$ items that contain $K$ successes in total | $\{\max(0,n+K-N),\dots,\min(n,K)\}$ | $n\frac{K}{N}$ | $n\frac{K(N-K)(N-n)}{N^2(N-1)}$ | $\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$ |
| Negative Binomial | $X \sim \text{NegBin}(r,p)$ for $r \in \mathbb{N}, p \in [0,1]$ | Sum of $r$ iid $Geo(p)$ rvs. # of independent flips until $r^{\text{th}}$ head with $P(head)=p$ | $\{r,r+1,\dots\}$ | $\frac{r}{p}$ | $\frac{r(1-p)}{p^2}$ | $\binom{k-1}{r-1}p^r(1-p)^{k-r}$ |
| Multinomial | $\mathbf{X} \sim \text{Mult}_r(n,\mathbf{p})$ for $r,n \in \mathbb{N}$ and $\mathbf{p}=(p_1,p_2,\dots,p_r)$, $\sum_{i=1}^r p_i = 1$ | Generalization of the Binomial distribution, $n$ trials with $r$ categories each with probability $p_i$ | $k_i \in \{0,\dots,n\}$, $i \in \{1,\dots,r\}$ and $\Sigma k_i = n$ | $\mathbb{E}[\mathbf{X}] = n\mathbf{p} = \begin{bmatrix} np_1 \\ \vdots \\ np_r \end{bmatrix}$ | $Var(X_i) = np_i(1-p_i)$ $Cov(X_i,X_j) = -np_ip_j, i \ne j$ | $\binom{n}{k_1,\dots,k_r}\prod_{i=1}^r p_i^{k_i}$ |
| Multivariate Hypergeometric | $\mathbf{X} \sim \text{MVHG}_r(N,\mathbf{K},n)$ for $r,n \in \mathbb{N}$, $\mathbf{K} \in \mathbb{N}^r$ and $N = \sum_{i=1}^r K_i$ | Generalization of the Hypergeometric distribution, $n$ draws from $r$ categories each with $K_i$ successes (w/out replacement) | $k_i \in \{0,\dots,K_i\}$, $i \in \{1,\dots,r\}$ and $\Sigma k_i = n$ | $\mathbb{E}[\mathbf{X}] = n\frac{\mathbf{K}}{N} = \begin{bmatrix} n\frac{K_1}{N} \\ \vdots \\ n\frac{K_r}{N} \end{bmatrix}$ | $Var(X_i) = n\frac{K_i}{N}\cdot\frac{N-K_i}{N}\cdot\frac{N-n}{N-1}$ $Cov(X_i,X_j) = -n\frac{K_i}{N}\frac{K_j}{N}\cdot\frac{N-n}{N-1}, i \ne j$ | $\frac{\prod_{i=1}^r \binom{K_i}{k_i}}{\binom{N}{n}}$ |

**University of Washington CSE312** — Alex Tsun & Mitchell Estberg

**Continuous Distributions**

| Distribution | Parameters | Possible Description | Range $\Omega_X$ | $\mathbb{E}[X]$ | Var(X) | PDF $f_X(x)$ for $x \in \Omega_X$ | CDF ($\mathbf{F_X(x)} = \mathbb{P}(\mathbf{X} \le \mathbf{x})$) |
|---|---|---|---|---|---|---|---|
| Uniform | $X \sim \text{Unif}(a,b)$ | Equally likely to be | | $a+b$ | $(b-a)^2$ | $1$ | $0$ if $x < a$ |