Linearity of Expectation

CSE 312 Su23
Lecture 8

LOTUS

Expectation

Expectation

The "expectation" (or "expected value") of a random variable X is:

$$\mathbb{E}[X] = \sum_k k \cdot \mathbb{P}(X = k)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

How do we compute $\mathbb{E}[g(\mathrm{X})]$?

Law of the Unconscious Statistician (LOTUS)

Given a discrete RV $X: \Omega \to \mathbb{R}$, then the expected value of g(X) is

$$\mathbb{E}[g(\mathrm{X})] = \sum_{x \in \Omega_X} g(x) \cdot \mathbb{P}(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Example

Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

Let X be the result of the die roll.

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Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

Let X be the result of the die roll. Then, our winnings are $10X^3$.

$\mathbb{E}[10X^3] = \sum_{x=1}^{6} 10x^3 \cdot \mathbb{P}(X = x)$

$= 10 \cdot \frac{1}{6} + 10 \cdot 2^3 \cdot \frac{1}{6} + 10 \cdot 3^3 \cdot \frac{1}{6} + 10 \cdot 4^3 \cdot \frac{1}{6} + 10 \cdot 5^3 \cdot \frac{1}{6} + 10 \cdot 6^3 \cdot \frac{1}{6}$

Independence of RVs

Independence

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Two events A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

Equivalently,

If $\mathbb{P}(B) \neq 0,\ \mathbb{P}(A|B) = \mathbb{P}(A)$.

If $\mathbb{P}(A) \neq 0,\ \ \mathbb{P}(B|A) = \mathbb{P}(B)$

Independence for 3 or more events

For three or more events, we need two kinds of independence

Pairwise Independence

Events $A_1, A_2, \ldots, A_n$ are pairwise independent if
$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j)$ for all $i, j,\ i \neq j$

Mutual Independence

Events $A_1, A_2, \ldots, A_n$ are mutually independent if
$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$
for every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n\}$.

Independence of Random Variables

That's for events… what about random variables?

Independence (of random variables)

X **and** Y **are independent if for all** k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of ∩ symbol.

Independence of Random Variables

The "for all values" is important.

Define $S =$"the sum of two dice" and $R =$"the value of the red die"

$$\mathbb{P}(S = 7, R = 5) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}(S = 7)\mathbb{P}(R = 5)$$

$(S = 7, R = 5)$: {(5,2)}

$(S = 7)$: {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

$(R = 5)$: {(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

Independence of Random Variables

The “for all values” is important.

Define $S =$“the sum of two dice” and $R =$“the value of the red die”

But,

$$\mathbb{P}(S = 2, R = 5) = 0 \neq \frac{1}{36} \cdot \frac{1}{6} = \mathbb{P}(S = 2)\mathbb{P}(R = 5) \text{ (for example)}$$

$(S = 2)$: {(1,1)}

$(R = 5)$: {(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

NOT independent.

Mutual Independence for RVs

A little simpler to write down than for events

Mutual Independence (of random variables)

$X_1, X_2, \dots, X_n$ **are mutually independent if for all** $x_1, x_2, \dots, x_n$

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

DON'T need to check all subsets for random variables…

But you do need to check all values (all possible x_i) still.

Why do we care?

Independent random variables have some special properties! We'll look at a lot of these in the next few days…

Linearity of Expectation

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Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X}+\boldsymbol{Y}]=\mathbb{E}[\boldsymbol{X}]+\mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1+X_2+\cdots+X_n]=\mathbb{E}[X_1]+\mathbb{E}[X_2]+\cdots+\mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y:

$$\mathbb{E}[\boldsymbol{X} + \boldsymbol{Y}] = \mathbb{E}[\boldsymbol{X}] + \mathbb{E}[\boldsymbol{Y}]$$

Note: X and Y do not have to be independent

Proof:

$\mathbb{E}[X + Y] = \Sigma_\omega P(\omega)(X(\omega) + Y(\omega))$

$= \Sigma_\omega \ P(\omega)X(\omega) + P(\omega)Y(\omega)$

$= \Sigma_\omega P(\omega)X(\omega) + \Sigma_\omega P(\omega)Y(\omega)$

$= \mathbb{E}[X] + \mathbb{E}[Y]$

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Constants are also fine:

For real numbers a, b, c

$$\mathbb{E}[aX+bY+c]=a\mathbb{E}[X]+b\mathbb{E}[Y]+c$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

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- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

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Let X be the r.v. representing the total number of heads

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{4} & \text{if } x = 2 \end{cases}$$

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$$\mathbb{E}[X] = \Sigma_k \mathbb{P}(X = k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let Y be the r.v. representing the total number of heads.

Make a prediction --- what should $\mathbb{E}[Y]$ be?

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let Y be the r.v. representing the total number of heads.

$$\mathbb{E}[Y] = \textstyle\sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Ok, but what actually is it?
I don't have intuition for this formula.

Repeated Coin Tosses

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$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Binomial Theorem

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[Y] = \sum_{k=0}^{n} k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} \ldots$$

$$= \sum_{k=1}^{n} k \cdot \binom{n}{k} \ldots$$

$$\ldots \binom{\ldots}{k-1} p^{k} (1-p)^{n-k}$$

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Binomial Theorem!

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No one wants to do proofs like this every time!

Indicator Random Variables

For any event A, we can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event A occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(A)$$
$$\mathbb{P}(X = 0) = 1 - \mathbb{P}(A)$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the ith coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_i = 1) = p$$
$$\mathbb{P}(X_i = 0) = 1 - p$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

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$$X = \sum_{i=1}^{n} X_i$$

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$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \mathrm{n} \cdot \mathbb{E}[X_i] = np$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. <u>Decompose</u>: Finding the right way to decompose the random variable into sum of simple random variables
$$X = X_1 + X_2 + \cdots + X_n$$
2. <u>LOE</u>: Apply Linearity of Expectation
$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$
3. <u>Conquer</u>: Compute the expectation of each X_i

Often, X_i are indicator random variables.

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

<u>Decompose:</u>

<u>LOE:</u>

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Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

<u>Decompose:</u> Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person i, j have the same bithday} \\ 0 & \text{otherwise} \end{cases} \qquad X = \sum_{1 \le i < j \le m} X_{ij}$$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}\left[\Sigma_{1 \le i < j \le m} X_{ij}\right] = \Sigma_{1 \le i < j \le m} \mathbb{E}\left[X_{ij}\right]$$

<u>Conquer:</u>

$$\mathbb{E}\left[X_{ij}\right] = P\left(X_{ij} = 1\right) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}\left[X_{ij}\right] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and n-1 (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

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<u>Decompose:</u> Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person i sits infront of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \Sigma_{i=1}^{n} X_i$

<u>LOE:</u>

$$\mathbb{E}[X] = \mathbb{E}[\Sigma_{i=1}^{n} X_i] = \Sigma_{i=1}^{n} \mathbb{E}[X_i]$$

<u>Conquer:</u>

These X_i are not independent!
That's ok!!

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<u>Conquer:</u>

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \qquad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$