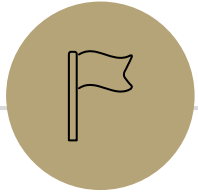


Linearity of Expectation

CSE 312 Su23
Lecture 8



LOTUS

Expectation

Expectation

The “expectation” (or “expected value”) of a random variable X is:

$$\mathbb{E}[X] = \sum_k k \cdot \mathbb{P}(X = k)$$

Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

How do we compute $\mathbb{E}[g(X)]$?

Law of the Unconscious Statistician (LOTUS)

Given a discrete RV $X: \Omega \rightarrow \mathbb{R}$, then the expected value of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot \mathbb{P}(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Example

Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

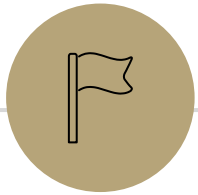
Let X be the result of the die roll.

Example

Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

Let X be the result of the die roll. Then, our winnings are $10X^3$.

$$\begin{aligned}\mathbb{E}[10X^3] &= \sum_{x=1}^6 10x^3 \cdot \mathbb{P}(X = x) \\ &= 10 \cdot \frac{1}{6} + 10 \cdot 2^3 \cdot \frac{1}{6} + 10 \cdot 3^3 \cdot \frac{1}{6} + 10 \cdot 4^3 \cdot \frac{1}{6} + 10 \cdot 5^3 \cdot \frac{1}{6} + 10 \cdot 6^3 \cdot \frac{1}{6}\end{aligned}$$



Independence of RVs

Independence

Independence

Two events A, B are independent if
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

Equivalently,

If $\mathbb{P}(B) \neq 0$, $\mathbb{P}(A|B) = \mathbb{P}(A)$.

If $\mathbb{P}(A) \neq 0$, $\mathbb{P}(B|A) = \mathbb{P}(B)$

Independence for 3 or more events

For three or more events, we need two kinds of independence

Pairwise Independence

Events $A_1, A_2, \dots, A_n$ are pairwise independent if
$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j) \text{ for all } i, j, i \neq j$$

Mutual Independence

Events $A_1, A_2, \dots, A_n$ are mutually independent if
$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$$

for every subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, n\}$.

Independence of Random Variables

That's for events... what about random variables?

Independence (of random variables)

X and Y are independent if for all k, ℓ
$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of $\cap$ symbol.

Independence of Random Variables

The “for all values” is important.

Define S = “the sum of two dice” and R = “the value of the red die”

$$\mathbb{P}(S = 7, R = 5) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}(S = 7)\mathbb{P}(R = 5)$$

$(S = 7, R = 5): \{(5,2)\}$

$(S = 7): \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$(R = 5): \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

Independence of Random Variables

The “for all values” is important.

Define S = “the sum of two dice” and R = “the value of the red die”

But,

$$\mathbb{P}(S = 2, R = 5) = 0 \neq \frac{1}{36} \cdot \frac{1}{6} = \mathbb{P}(S = 2)\mathbb{P}(R = 5) \text{ (for example)}$$

$(S = 2): \{(1,1)\}$

$(R = 5): \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

NOT independent.

Mutual Independence for RVs

A little simpler to write down than for events

Mutual Independence (of random variables)

$X_1, X_2, \dots, X_n$ are mutually independent if for all $x_1, x_2, \dots, x_n$

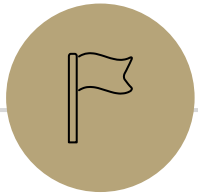
$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1)\mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$

DON'T need to check all subsets for random variables...

But you do need to check all values (all possible x_i) still.

Why do we care?

Independent random variables have some special properties! We'll look at a lot of these in the next few days...



Linearity of Expectation

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Proof:

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_{\omega} P(\omega)(X(\omega) + Y(\omega)) \\ &= \sum_{\omega} P(\omega)X(\omega) + P(\omega)Y(\omega) \\ &= \sum_{\omega} P(\omega)X(\omega) + \sum_{\omega} P(\omega)Y(\omega) \\ &= \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

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Constants are also fine:

For real numbers a, b, c

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$
- How many fish do both of you bring on an average day?

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$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

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Let X be the r.v. representing the total number of heads

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{4} & \text{if } x = 2 \end{cases}$$

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$$\mathbb{E}[X] = \sum_k \mathbb{P}(X = k) \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let Y be the r.v. representing the total number of heads.

Make a prediction --- what should $\mathbb{E}[Y]$ be?

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let Y be the r.v. representing the total number of heads.

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k}\end{aligned}$$

Ok, but what actually is it?
I don't have intuition for this
formula.

Repeated Coin Tosses

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$$\begin{aligned}\mathbb{E}[Y] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\ &= np(p + (1-p))^{n-1} = np\end{aligned}$$

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

Binomial Theorem

$$\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

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Binomial Theorem!

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No one wants to do proofs like this every time!

Indicator Random Variables

For any event A , we can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(A) \\ \mathbb{P}(X = 0) &= 1 - \mathbb{P}(A) \end{aligned}$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\mathbb{P}(X_i = 1) &= p \\ \mathbb{P}(X_i = 0) &= 1 - p\end{aligned}$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

Repeated Coin Tosses

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$$X = \sum_{i=1}^n X_i$$

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By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i] = n \cdot \mathbb{E}[X_i] = np$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + X_2 + \cdots + X_n$$

2. LOE: Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. Conquer: Compute the expectation of each X_i

Often, X_i are indicator random variables.

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

LOE:

Conquer:

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Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person } i, j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases} \quad X = \sum_{1 \leq i < j \leq m} X_{ij}$$

LOE:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{1 \leq i < j \leq m} X_{ij}\right] = \sum_{1 \leq i < j \leq m} \mathbb{E}[X_{ij}]$$

Conquer:

$$\begin{aligned} \mathbb{E}[X_{ij}] &= P(X_{ij} = 1) = \frac{365}{365 \cdot 365} = \frac{1}{365} \\ \mathbb{E}[X] &= \binom{m}{2} \cdot \mathbb{E}[X_{ij}] = \binom{m}{2} \cdot \frac{1}{365} \end{aligned}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

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Decompose: Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ sits in front of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \sum_{i=1}^n X_i$

LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

These X_i are not independent!
That's ok!!

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LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \quad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$