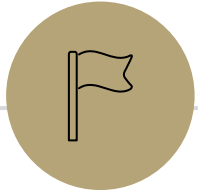


Linearity of Expectation

CSE 312 Su23
Lecture 8



LOTUS

Expectation

Expectation

The “expectation” (or “expected value”) of a random variable X is:

$$\mathbb{E}[X] = \sum_k k \cdot \mathbb{P}(X = k)$$


Intuition: The weighted average of values X could take on.

Weighted by the probability you actually see them.

How do we compute $\mathbb{E}[g(X)]$?

Law of the Unconscious Statistician (LOTUS)

Given a discrete RV $X: \Omega \rightarrow \mathbb{R}$, then the expected value of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot \mathbb{P}(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

$$\mathbb{E}[x^2] = \sum_{x \in \Omega_X} x^2 \cdot \mathbb{P}(X = x)$$

Example

Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

Let X be the result of the die roll.

$$Y = 10X^3$$

LOTUS



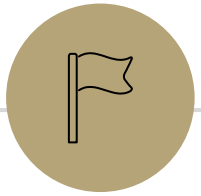
$$E[Y] = E[10X^3] = \sum_{x \in \mathcal{X}} 10x^3 \cdot P(X=x)$$

Example

Suppose we roll a fair, 6-sided die. You win the cube of the number rolled, times 10, in dollars. How much do you expect to win?

Let X be the result of the die roll. Then, our winnings are $10X^3$.

$$\begin{aligned}\mathbb{E}[10X^3] &= \sum_{x=1}^6 10x^3 \cdot \mathbb{P}(X = x) \\ &= 10 \cdot \frac{1}{6} + 10 \cdot 2^3 \cdot \frac{1}{6} + 10 \cdot 3^3 \cdot \frac{1}{6} + 10 \cdot 4^3 \cdot \frac{1}{6} + 10 \cdot 5^3 \cdot \frac{1}{6} + 10 \cdot 6^3 \cdot \frac{1}{6}\end{aligned}$$



Independence of RVs

Independence

Independence

Two events A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$


You'll sometimes see this called "statistical independence" to emphasize that we're talking about probabilities (not, say, physical interactions).

Equivalently,

If $\mathbb{P}(B) \neq 0$, $\mathbb{P}(A|B) = \mathbb{P}(A)$.

If $\mathbb{P}(A) \neq 0$, $\mathbb{P}(B|A) = \mathbb{P}(B)$

Independence for 3 or more events

For three or more events, we need two kinds of independence

✦ Pairwise Independence

Events $A_1, A_2, \dots, A_n$ are pairwise independent if

$$\underbrace{\mathbb{P}(A_i \cap A_j)} = \underbrace{\mathbb{P}(A_i)} \cdot \underbrace{\mathbb{P}(A_j)} \text{ for all } i, j, i \neq j$$

✦ Mutual Independence

Events $A_1, A_2, \dots, A_n$ are mutually independent if

$$\underbrace{\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k})}_{\text{for every subset } \{i_1, i_2, \dots, i_k\} \text{ of } \{1, 2, \dots, n\}.} = \underbrace{\mathbb{P}(A_{i_1}) \cdot \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})}$$

Independence of Random Variables

That's for events... what about random variables?

Independence (of random variables)

X and Y are independent if for all k, ℓ

$$\mathbb{P}(X = k, Y = \ell) = \mathbb{P}(X = k)\mathbb{P}(Y = \ell)$$

We'll often use commas instead of $\wedge$ symbol.

Independence of Random Variables

The "for all values" is important.

Define S = "the sum of two dice" and R = "the value of the red die"

$$\mathbb{P}(S = 7, R = 5) = \frac{1}{36} = \frac{1}{6} \cdot \frac{1}{6} = \mathbb{P}(S = 7)\mathbb{P}(R = 5)$$

$(S = 7, R = 5): \{(5, 2)\}$

$(S = 7): \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

$(R = 5): \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

Independence of Random Variables

The “for all values” is important.

Define S = “the sum of two dice” and R = “the value of the red die”

But,

$$\mathbb{P}(S = 2, R = 5) = 0 \neq \frac{1}{36} \cdot \frac{1}{6} = \mathbb{P}(S = 2)\mathbb{P}(R = 5) \text{ (for example)}$$

$(S = 2): \{(1,1)\}$

$(R = 5): \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$

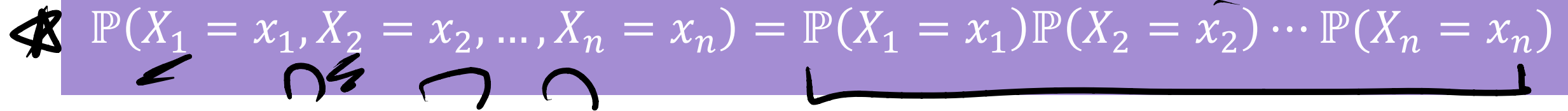
NOT independent.

Mutual Independence for RVs

A little simpler to write down than for events

Mutual Independence (of random variables)

$X_1, X_2, \dots, X_n$ are mutually independent if for all $x_1, x_2, \dots, x_n$

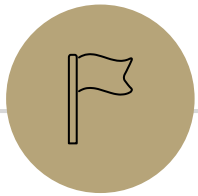
$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \mathbb{P}(X_2 = x_2) \cdots \mathbb{P}(X_n = x_n)$$


DON'T need to check all subsets for random variables...

But you do need to check all values (all possible x_i) still.

Why do we care?

Independent random variables have some special properties! We'll look at a lot of these in the next few days...



Linearity of Expectation

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Extending this to n random variables, $X_1, X_2, \dots, X_n$

$$\star \mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n]$$

This can be proven by induction.

Linearity of Expectation - Proof

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Proof:

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_{\omega} P(\omega)(X(\omega) + Y(\omega)) && \text{defn } \mathbb{E}[X] \\&= \sum_{\omega} P(\omega)X(\omega) + \sum_{\omega} P(\omega)Y(\omega) \\&= \underbrace{\sum_{\omega} P(\omega)X(\omega)}_{\mathbb{E}[X]} + \underbrace{\sum_{\omega} P(\omega)Y(\omega)}_{\mathbb{E}[Y]} \\&= \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

$\mathbb{E}[X] = \sum_{\omega} P(\omega)X(\omega)$

Linearity of Expectation

Linearity of Expectation

For any two random variables X and Y :

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

Note: X and Y do not have to be independent

Constants are also fine:

For real numbers a, b, c

$$\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + \mathbb{E}[c]$$

$\downarrow$

$$\mathbb{E}[X^2 + Y] = \mathbb{E}[X^2] + \mathbb{E}[Y]$$

Fishy Business

Say you and your friend go fishing everyday.

- You catch X fish, with $\mathbb{E}[X] = 3$ ★
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$ ★
- How many fish do both of you bring on an average day?

$$Z = X + Y$$

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] \stackrel{\text{LoE}}{=} \mathbb{E}[X] + \mathbb{E}[Y] \\ = 3 + 7 = 10$$

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Let Z be the r.v. representing the total number of fish you both catch

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$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

- You can sell each for \$10 per fish, but you need \$15 (total) for expenses.
What is your average profit?

$$A = 10Z - 15$$

$$\mathbb{E}[A] = \mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15$$

Fishy Business

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- You catch X fish, with $\mathbb{E}[X] = 3$
- Your friend catches Y fish, with $\mathbb{E}[Y] = 7$

- How many fish do both of you bring on an average day?

Let Z be the r.v. representing the total number of fish you both catch

$$\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 7 = 10$$

- You can sell each for \$10 per fish, but you need \$15 (total) for expenses. What is your average profit?

$$\mathbb{E}[10Z - 15] = 10\mathbb{E}[Z] - 15 = 100 - 15 = 85$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Let X be the r.v. representing the total number of heads

$$p_X(x) = \begin{cases} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{cases}$$

$$\begin{aligned} \text{if } x = 0 & \quad \{TT\} \\ \text{if } x = 1 & \quad \{HT, TH\} \\ \text{if } x = 2 & \quad \{HH\} \end{aligned}$$

Coin Tosses

If we flip a coin twice, what is the expected number of heads that come up?

Let X be the r.v. representing the total number of heads

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 0 \\ \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{4} & \text{if } x = 2 \end{cases}$$

$$\mathbb{E}[X] = \underbrace{\sum_k}_{\text{Σ}} \underbrace{\mathbb{P}(X = k)}_{\text{P}} \cdot k = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?



Let Y be the r.v. representing the total number of heads.

Make a prediction --- what should $\mathbb{E}[Y]$ be?

$$\frac{1}{2} \cdot 2 = 1$$

$$np$$

Repeated Coin Tosses



Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

Let Y be the r.v. representing the total number of heads.

$$\begin{aligned} \mathbb{E}[Y] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \end{aligned}$$

Handwritten annotations: An arrow points from the first k in the first sum to the second k in the second sum. A bracket under the first sum is crossed out with a squiggle. A bracket under the second sum is also crossed out with a squiggle. Above the second sum, three arrows point down to the terms k , $\binom{n}{k}$, and p^k , and a squiggle is under $(1-p)^{n-k}$.

Ok, but what actually is it?
I don't have intuition for this
formula.

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^n n \cdot \binom{n-1}{k-1} p^k (1-p)^{n-k} \\ &= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\ &= np(p + (1-p))^{n-1} = \underline{np}\end{aligned}$$

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$



Binomial Theorem

$$\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

Repeated Coin Tosses

Now what if the probability of flipping a heads was p and that we wanted to find the total number of heads flipped when we flip the coin n times?

$$\mathbb{E}[Y] = \sum_{k=0}^n k \cdot \mathbb{P}(Y = k) = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^n k \binom{n}{k} p^k (1-p)^{n-k}$$

$$= np \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}$$

$$= np(p + (1-p))^{n-1} = np$$

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

Binomial Theorem!

We did it! And all it took was a clever application of the binomial theorem, setup by a very non-obvious application of an obscure combinatorial identity. Ezpz.

No one wants to do proofs like this every time!

Indicator Random Variables

For any event A , we can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 \\ 0 \end{cases} \begin{matrix} \longrightarrow \\ \longrightarrow \end{matrix} \begin{matrix} \text{if event } A \text{ occurs} \\ \text{otherwise} \end{matrix}$$

$$\Omega_{X_A} = \{0, 1\}$$

$$\mathbb{P}(X=x)$$

$$p_X(x) = \begin{cases} \mathbb{P}(A) & \text{if } x = 1 \\ 1 - \mathbb{P}(A) & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X=2)$$

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

What indicators can we define? What 'Booleans' have enough information to combine (add) and solve the problem?

H H T T H

1 1 0 0 1 = 3

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i]$$

X : total n flips

$X_i = \begin{cases} 1 & \text{if } i\text{th flip H} \\ 0 & \text{o.w.} \end{cases}$

$$\downarrow \\ X = \sum_{i=1}^n X_i$$

Repeated Coin Tosses

$$\sum_{i=1}^n \mathbb{E}[X_i] = \sum_{i=1}^n p = n \cdot p$$

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

Let X be the total number of heads

Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if the } i\text{th coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbb{P}(X_i = 1) &= p \\ \mathbb{P}(X_i = 0) &= 1 - p \end{aligned}$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

$$\Omega_{X_i} = \{0, 1\} \quad \text{def } \mathbb{E}[X]$$

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

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$$X = \sum_{i=1}^n X_i$$

Repeated Coin Tosses

The probability of flipping a heads is p and we want to find the total number of heads flipped when we flip the coin n times?

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$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p$$

By Linearity of Expectation,

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i] = n \cdot \mathbb{E}[X_i] = np$$

Computing complicated expectations

We often use these three steps to solve complicated expectations

1. Decompose: Finding the right way to decompose the random variable into sum of simple random variables

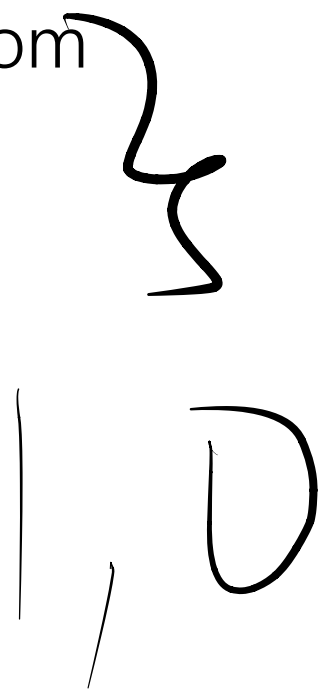
$$X = X_1 + X_2 + \cdots + X_n$$

2. LOE: Apply Linearity of Expectation

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]$$

3. Conquer: Compute the expectation of each X_i

Often, X_i are indicator random variables.

A large handwritten curly brace on the right side of the slide, spanning the first three steps. Below it, there is a vertical line followed by a circle, resembling a stylized 'D' or a list marker.

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Property:

Decompose:

LOE:

Conquer:

$$x_i = \begin{cases} 1 & \text{if ith pair same bday} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{\binom{m}{2}} x_i$$

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{\binom{m}{2}} X_i\right]$$

$\underbrace{\frac{1}{365} + \dots + \frac{1}{365}}_{\binom{m}{2}}$

LOE:

$$= \sum_{i=1}^{\binom{m}{2}} \mathbb{E}[X_i]$$

Conquer:

$$= \sum_{i=1}^{\binom{m}{2}} \frac{1}{365} \xrightarrow{\text{(next slide)}} \binom{m}{2} \frac{1}{365}$$

Pairs with the same birthday

$$X_i = \begin{cases} 1 & \text{if the pair is same} \\ 0 & \text{otherwise} \end{cases}$$

In a class of m students, on average how many pairs of people have the same birthday?

Decompose:

$$E[X_i] = 0 \cdot P(X_i=0) + 1 \cdot P(X_i=1)$$

$$= P(X_i=1)$$

LOE:

$$= P(\text{same birthday})$$

Conquer:

$$= \frac{365 \cdot 1}{365 \cdot 365} = \frac{1}{365}$$

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday?

Decompose: Let X be the number of pairs with the same birthday

Define X_{ij} as follows:

$$X_{ij} = \begin{cases} 1 & \text{if person } i, j \text{ have the same birthday} \\ 0 & \text{otherwise} \end{cases}$$

LOE:

$$X = \sum_{1 \leq i < j \leq m} X_{ij}$$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{1 \leq i < j \leq m} X_{ij}\right] = \sum_{1 \leq i < j \leq m} \mathbb{E}[X_{ij}]$$

Conquer:

$$\mathbb{E}[X_{ij}] = P(X_{ij} = 1) = \frac{365}{365 \cdot 365} = \frac{1}{365}$$

$$\mathbb{E}[X] = \binom{m}{2} \cdot \mathbb{E}[X_{ij}] = \binom{m}{2} \cdot \frac{1}{365}$$

Rotating the table $1 \leq i \leq n$

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

$$X_i = \begin{cases} 1 \\ 0 \end{cases}$$

i th person in front of name tag

Rotating the table

$$\frac{n}{n-1}$$

n people are sitting around a circular table. Nobody is sitting in front of their own name tag. Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

Let X be the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

Decompose:

LOE:

Conquer:

$$x_i = \begin{cases} 1 & \text{ith person in front} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{i=1}^n x_i$$

$$\mathbb{E}[x_i] = \mathbb{P}(x_i = 1) = \frac{1}{n-1}$$

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}\left[\sum_{i=1}^n x_i\right] \\ &= \sum_{i=1}^n \mathbb{E}[x_i] = \sum_{i=1}^n \frac{1}{n-1} \end{aligned}$$

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

X is the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

Decompose: Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ sits in front of their own name tag} \\ 0 & \text{otherwise} \end{cases}$$

Note: $X = \sum_{i=1}^n X_i$

LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

These X_i are not independent!
That's ok!!

Rotating the table

n people are sitting around a circular table. There is a name tag in each place. Nobody is sitting in front of their own name tag.

Rotate the table by a random number k of positions between 1 and $n-1$ (equally likely)

X is the number of people that end up in front of their own name tag. Find $\mathbb{E}[X]$.

Decompose: Define X_i as follows:

$$X_i = \begin{cases} 1 & \text{if person } i \text{ sits in front of their own name tag} \\ 0 & \text{otherwise} \end{cases} \quad X = \sum_{i=1}^n X_i$$

LOE:

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathbb{E}[X_i]$$

Conquer:

$$\mathbb{E}[X_i] = P(X_i = 1) = \frac{1}{n-1} \quad \mathbb{E}[X] = n \cdot \mathbb{E}[X_i] = \frac{n}{n-1}$$