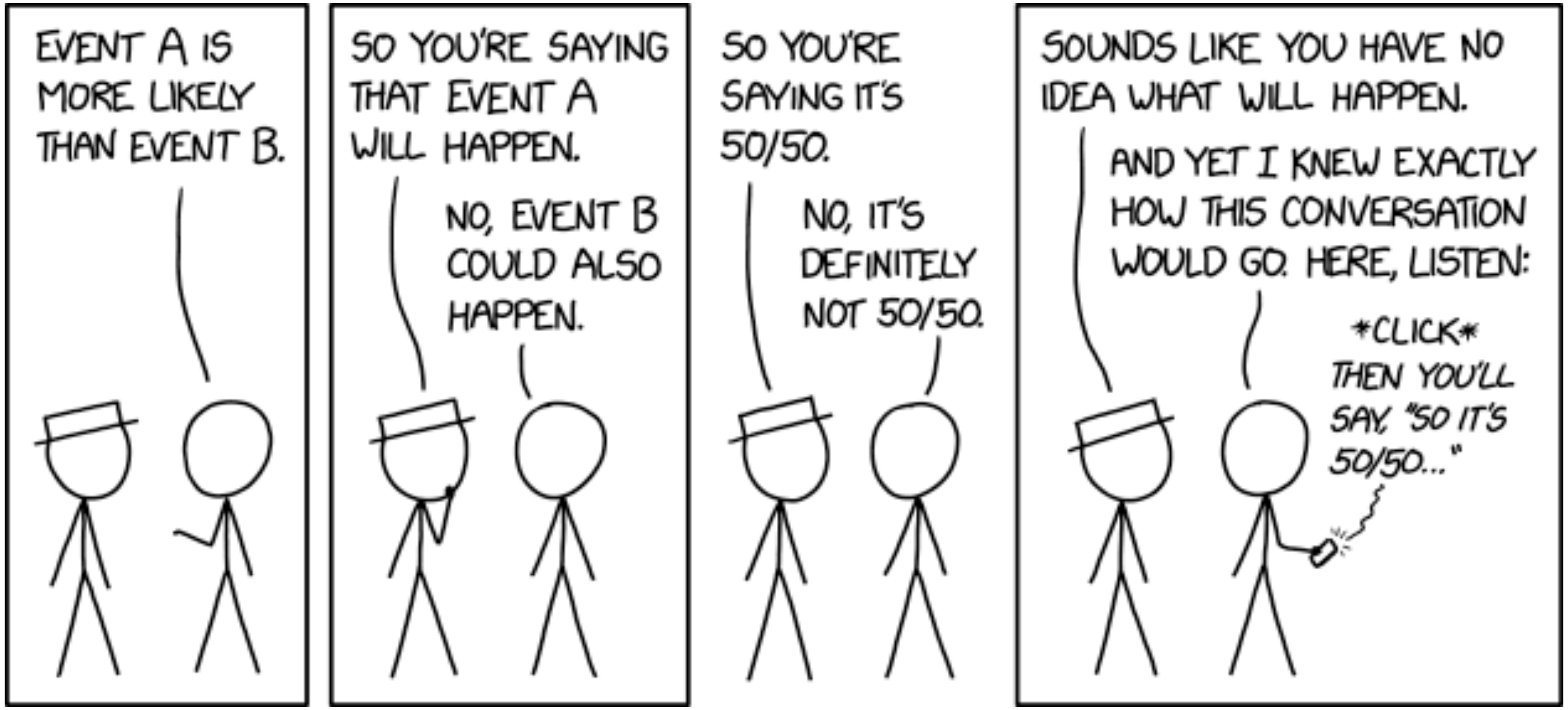

https://xkcd.com/2370/

Discrete Probability

CSE 312 Su23
Lecture 4

Announcements

- HW1 due today at **11.00pm**
- HW2 out today evening, due next Wednesday at 11.00pm

Today

Starting today, we get to calculate probabilities!

Mostly notation and vocabulary today.

Probability

In the real-world, many things are not certain to happen.

The classic examples:

when a coin is flipped, we are uncertain if the **outcome** will be H or T

when a dice is rolled, we are uncertain if the **outcome** will be 1, 2, 3, 4, 5 or 6

Probability is a way of **quantifying** our uncertainty when more than one outcome is possible.

Probability

A note: when working with these "real-world" examples, we're going to state some assumptions:

- We can flip a coin, and each face is equally likely to come up
- We can roll a die, and every number is equally likely to come up
- We can shuffle a deck of cards so that every ordering is equally likely.

Sample Space

Omega

Sample Space

A sample space Ω is the set of all possible outcomes of an experiment.

What is Ω?
Omega

What is an outcome?
A possible **result** of an experiment.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \{HH, HT, TH, TT\}$

For rolling a 6-sided die: $\Omega =$

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A sample space Ω is the set of all possible outcomes of an experiment.

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What is an outcome?
A possible **result** of an experiment.

Examples:

For a single coin flip, $\Omega = \{H, T\}$

For a series of two coin flips, $\Omega = \boxed{\{HH, HT, TH, TT\}}$ or $\{H, T\} \times \{H, T\}$

For rolling a 6-sided die: $\Omega = \{1,2,3,4,5,6\}$

Events

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

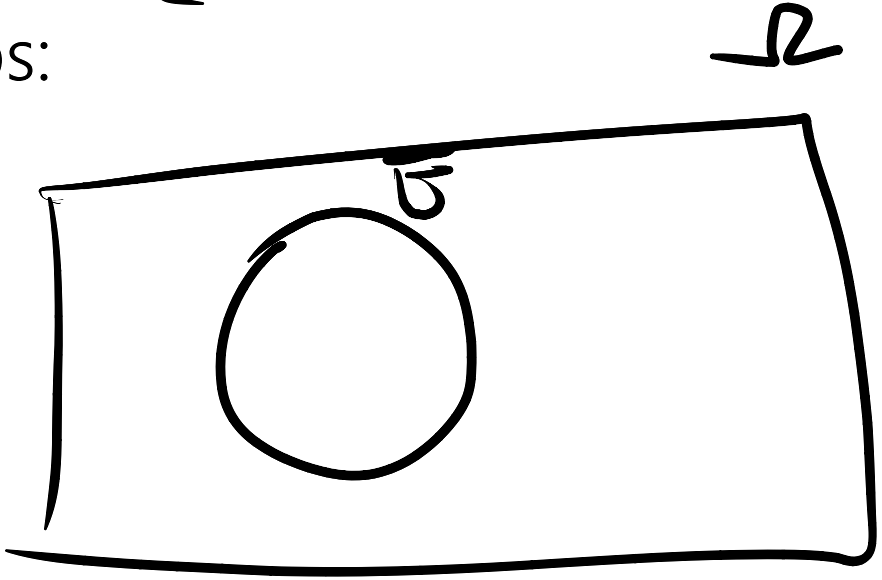
Examples:

Get at least one head among two coin flips:

E: $\{HH, HT, TH\}$

Get an even number on a die-roll:

$\{2, 4, 6\}$



Events

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An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

Examples:

Get at least one head among two coin flips ($E = \{HH, HT, TH\}$)

Get an even number on a die-roll ($E = \{2,4,6\}$).

Examples

Ω $|\Omega|$

I roll a blue 4-sided die and a red 4-sided die.

The table contains the sample space.

The event "the sum of the dice is even" is in gold
The event "the blue die is 1" is in green

Blue \ Red	D2=1	D2=2	D2=3	D2=4
D1=1	(1,1)	(1,2)	(1,3)	(1,4)
D1=2	(2,1)	(2,2)	(2,3)	(2,4)
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Examples

I roll a blue 4-sided die and a red 4-sided die.

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The event “the sum of the dice is even” is in gold
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	D2=1	D2=2	D2=3	D2=4
D1=1	$\boxed{(1,1)}$	(1,2)	$\boxed{(1,3)}$	(1,4)
D1=2	(2,1)	$\boxed{(2,2)}$	(2,3)	$\boxed{(2,4)}$
D1=3	$\boxed{(3,1)}$	(3,2)	$\boxed{(3,3)}$	(3,4)
D1=4	(4,1)	$\boxed{(4,2)}$	(4,3)	$\boxed{(4,4)}$

Examples

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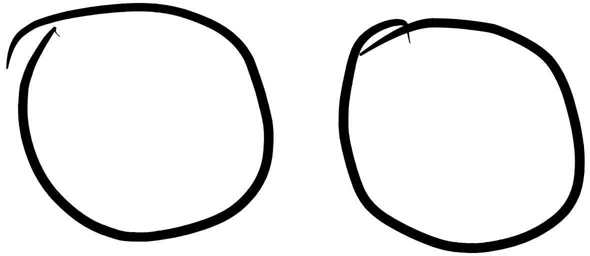
D1=3	(3,1)	(3,2)	(3,3)	(3,4)
D1=4	(4,1)	(4,2)	(4,3)	(4,4)

Mutually Exclusive Events



Two events E, F are mutually exclusive if they cannot happen simultaneously.

In notation, $E \cap F = \emptyset$ (i.e., they're disjoint subsets of the sample space).

For example, if $\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$

$E_1 =$"the coin came up heads"

$E_2 =$"the coin came up tails"

$E_3 =$"the die showed an even number"

Are E_1 and E_2 mutually exclusive?
Are E_1 and E_3 mutually exclusive?

Mutually Exclusive Events

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E_1 and E_2 are mutually exclusive.
E_1 and E_3 are not mutually exclusive.

Probability

Probability

A probability is a number between 0 and 1 describing how likely a particular outcome is.

We'll define a function

$\mathbb{P}: \Omega \to [0,1]$

$\mathbb{P}(\omega) =$

i.e., $\mathbb{P}$ takes an element of Ω as input and outputs the probability of the outcome.

We'll also use $\Pr[\omega]$, $\mathbb{P}(\omega)$ as notation.

Recall: Ω is the sample space or set of possible outcomes of an experiment. So $\mathbb{P}(\omega)$ for an outcome ω is the probability (a value between 0 and 1) of seeing that outcome.

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

$\mathbb{P}(H) = \frac{1}{2}$ $\quad 3/4$

$\mathbb{P}(T) = 1/2$ $\quad 1/4$

Example

Imagine we toss one coin.

Our sample space $\Omega = \{H, T\}$

What do you want $\mathbb{P}$ to be?

It depends on what we want to model

If the coin is fair $\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2}$.

But we also might have a biased coin: $\mathbb{P}(H) = .85$, $\mathbb{P}(T) = 0.15$.

Probability Space

$\frac{1}{2} + \frac{1}{2} = 1$ $\quad 3/4$

Probability Space

A (discrete) probability space is a pair $(\Omega, \mathbb{P})$ where:

Ω is the sample space

$\mathbb{P}: \Omega \to [0, 1]$ is the probability measure.

$\mathbb{P}$ satisfies:

- $\mathbb{P}(x) \geq 0$ for all x (*non-negativity*)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (*normalization*)
- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (*countable additivity*)

axioms

Probability Space

Flip a fair coin and roll a fair (6-sided) die.

$\Omega = \{H, T\} \times \{1,2,3,4,5,6\}$ $2 \times 6 = 12$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$ $1 \geq \frac{1}{12} \geq 0$

Is this a valid probability space?

$\mathbb{P}$ takes in elements of Ω and outputs numbers between 0 and 1

$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$

$12 \cdot \frac{1}{12} = 1$

There are 12 possible outcomes in the sample space, and each occurs with probability 1/12!

Events

$(H,1)$

$\Omega = \underbrace{\{H,T\}\times\{1,2,3,4,5,6\}}$

$\mathbb{P}(\omega) = \frac{1}{12}$ for every $\omega \in \Omega$

Event

An event $E \subseteq \Omega$ is a subset of possible outcomes (i.e. a subset of Ω)

So, what's the probability of seeing a heads?

$\{\underbrace{(H,1),\ldots,(H,6)}_{6}\}$

Seeing heads isn't an element of the sample space!

Define $\boxed{\mathbb{P}(E) = \sum_{\omega\in E}\mathbb{P}(\omega)}$

$\sum_{i=1}^{6}\frac{1}{12} = \frac{6}{12}$

This is a 'convenient' notation abuse: we extend $\mathbb{P}$ originally defined over outcomes, to be defined over sets of outcomes:
$\mathbb{P}(\omega) \to \mathbb{P}(\{\omega\})$

Probability Space

Probability Space

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- If $E, F \subseteq \Omega$ and $E \cap F = \emptyset$ then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (*countable additivity*)

Uniform Probability Space

The most common probability space is the **uniform** probability space, which is a probability space $(\Omega, \mathbb{P})$ such that for every $\omega \in \Omega$,

$$\mathbb{P}\left(\omega\right) = \frac{1}{|\Omega|}$$

For example,

fair coin: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{2}$

fair dice: for every $\omega \in \Omega$, $\mathbb{P}(\omega) = \frac{1}{6}$ $= \frac{1}{|\Omega|}$

Equally Likely Outcomes

We have a 6-sided fair die. What is the probability of rolling an even number?

Define E to be the event defined by the set of outcomes {2, 4, 6}.

Our goal is to calculate $\mathbb{P}(E)$.

$\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega) = \sum_{\omega \in \{2,4,6\}} \frac{1}{6} = \frac{3}{6}$

$$\sum_{i=1}^{|E|} \frac{1}{|\Omega|} = \boxed{\frac{|E|}{|\Omega|}}$$

Equally Likely Outcomes

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$$\mathbb{P}(E) = \sum_{\omega \in \mathrm{E}} \mathbb{P}(\omega) = \sum_{\omega \in \{2,4,6\}} \mathbb{P}(\omega) = \sum_{\omega \in \{2,4,6\}} \frac{1}{6} = \frac{3}{6} = \frac{|E|}{|\Omega|}$$

Equally likely outcomes

In a uniform probability space, for any event $E \subseteq \Omega$,

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Example

$\Omega : \{H, T\}^{100}$

$|\Omega| = 2^{100}$

Let your sample space be all possible outcomes of a sequence of 100 coin tosses. Assign the uniform measure to this sample space. What is the probability of the event "there are exactly 50 heads?

A. $\binom{100}{50}/2^{100}$ $\rightarrow \frac{|E|}{|\Omega|}$

B. $1/101$

C. $1/2$

D. $1/2^{50}$

E. There is not enough information in this problem.

Fill out the poll everywhere so Shreya can adjust her explanation
Go to pollev.com/312 and login with your UW identity

$|E| = \binom{100}{50}$

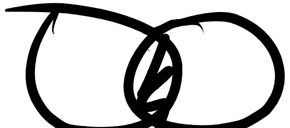
Axioms and Consequences

We wrote down 3 requirements (axioms) on probability measures

- $\mathbb{P}(x) \geq 0$ for all x (non-negativity)
- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$ (normalization)
- If E and F are mutually exclusive then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$ (countable additivity)

These lead quickly to these three corollaries $\bar{E} = \Omega - E$

- $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$ (complementation) $\mathbb{P}(\bar{E}) = 1 - \mathbb{P}(E)$
- If $E \subseteq F$, then $\mathbb{P}(E) \leq \mathbb{P}(F)$ (monotonicity)
- $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$ (inclusion-exclusion)



More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

$$\overbrace{\Omega : \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}}^{6\times6}$$

What is your sample space?

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? E: both dice are even

What is the probability?

$$\underbrace{\{2,4,6\}}_{3} \times \underbrace{\{2,4,6\}}_{\times 3}$$

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{9}{36}$$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What is your sample space? $\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$

What is your probability measure $\mathbb{P}$? $\mathbb{P}(\omega) = 1/36$ for all $\omega \in \Omega$

What is your event? $\{2,4,6\} \times \{2,4,6\}$

What is the probability? $3^2/6^2$

More Examples!

Suppose you roll two dice. Each die is fair and they don't affect each other. What is the probability of both dice being even?

What if we can't tell the dice apart and always put the dice in increasing order by value?

What is your sample space?

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6)$

$(3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)\}$

What is your probability measure $\mathbb{P}$?

$\mathbb{P}((x,y)) = 2/36$ if $x \neq y$, $\mathbb{P}(x,x) = 1/36$

What is your event? $\{(2,2), (4,4), (6,6), (2,4), (2,6), (4,6)\}$

What is the probability? $3 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} = \frac{9}{36}$

$\frac{|E|}{|\Omega|}$

Takeaways

There is often more than one sample space possible! But one is probably easier than the others.

Finding a sample space that will make the uniform measure correct will usually make finding the probabilities easier to calculate.

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space : $\binom{52}{2} = |\Omega|$ set of top two cards.

Probability Measure $\mathbb{P}(\omega) = 1/\binom{52}{2}$ $\omega \in \Omega$

Event

Probability

$|E|$ $\binom{13}{1}\binom{4}{2}$ $\frac{|E|}{|\Omega|}$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

$|\Omega| = 52 \times 51$

Sample Space: $\{(x, y): x$ and y are different cards $\}$

Probability Measure uniform measure $\mathbb{P}(\omega) = \frac{1}{52 \cdot 51}$

Event: all pairs with equal values

Probability $\frac{13 \cdot P(4,2)}{52 \cdot 51}$

$|E| = 13 \cdot P(4,2)$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space: Set of all orderings of all 52 cards $52!$

Probability Measure uniform measure $\mathbb{P}(\omega) = \frac{1}{52!}$

Event: all lists that start with two cards of the same value

Probability $\frac{13 \cdot P(4,2)\, \cancel{50!}}{\cancel{52!}} \rightarrow$

$52 \cdot 51$

Another Example

Suppose you shuffle a deck of cards so any arrangement is equally likely. What is the probability that the top two cards have the same value?

Sample Space

Probability Measure

Event

Probability

Takeaway

There’s often information you “don’t need” in your sample space.

It won’t give you the wrong answer.

But it sometimes makes for extra work/a harder counting problem,

Good indication: you cancelled A LOT of stuff that was common in the numerator and denominator.

Some Quick Observations

For discrete probability spaces (the kind we've seen so far)

$\mathbb{P}(E) = 0$ if and only if an event can't happen.

$\mathbb{P}(E) = 1$ if and only if an event is guaranteed (every outcome outside E has probability 0).

Same birthday

How many people do we need in this room to have at least a 50% chance of two people sharing a birthday?

A. 366

B. 365

C. 183

D. 23

E. 5

Fill out the poll everywhere because it's fun

Go to pollev.com/312

Birthday Paradox

With a probabilistic lens…

Let’s say we have *n* people in a room. Then, we are looking for the value of n such that the probability that 2 of the n people sharing a birthday is greater than 0.5.

So, what’s the probability that at least 2 of n people share a birthday?

Let’s maintain the following assumption: there are 365 possible birthdays, that are equally likely.

Suppose there are n people in a room. What is the probability that at least 2 of the n share a birthday, if there are 365 equally likely birthdays?

<2 ≤ 1

$\mathbb{P}(at\ least\ 2\ people\ in\ the\ room\ share\ a\ birthday)$

$1\ -\ \underbrace{\mathbb{P}(no\ one\ in\ the\ room\ shares\ a\ birthday)}$ (complementation)

Ω :

What's our sample space?

$\mathbb{P}(no\ one\ in\ the\ room\ shares\ a\ birthday)$

Our sample space is the set of possible birthdays that the n people can have.

- Birthdays can be repeated (i.e., are drawn 'with replacement')

$|\Omega| = number\ of\ possible\ birthday\ assignments = 365^n$

What's our sample space and event?

$\mathbb{P}(no\ one\ in\ the\ room\ shares\ a\ birthday)$

The event E is the set of birthday assignments such that no one shares a birthday with another.

$$|E| = \underline{P(365, n)} = \frac{365!}{(365-n)!}$$

Note that order matters here: since, our sample space accounts for order (when we assign birthdays with repetition).

Going back to our calculation…

Suppose there are n people in a room. What is the probability that at least 2 of the n share a birthday, if there are 365 equally likely birthdays?

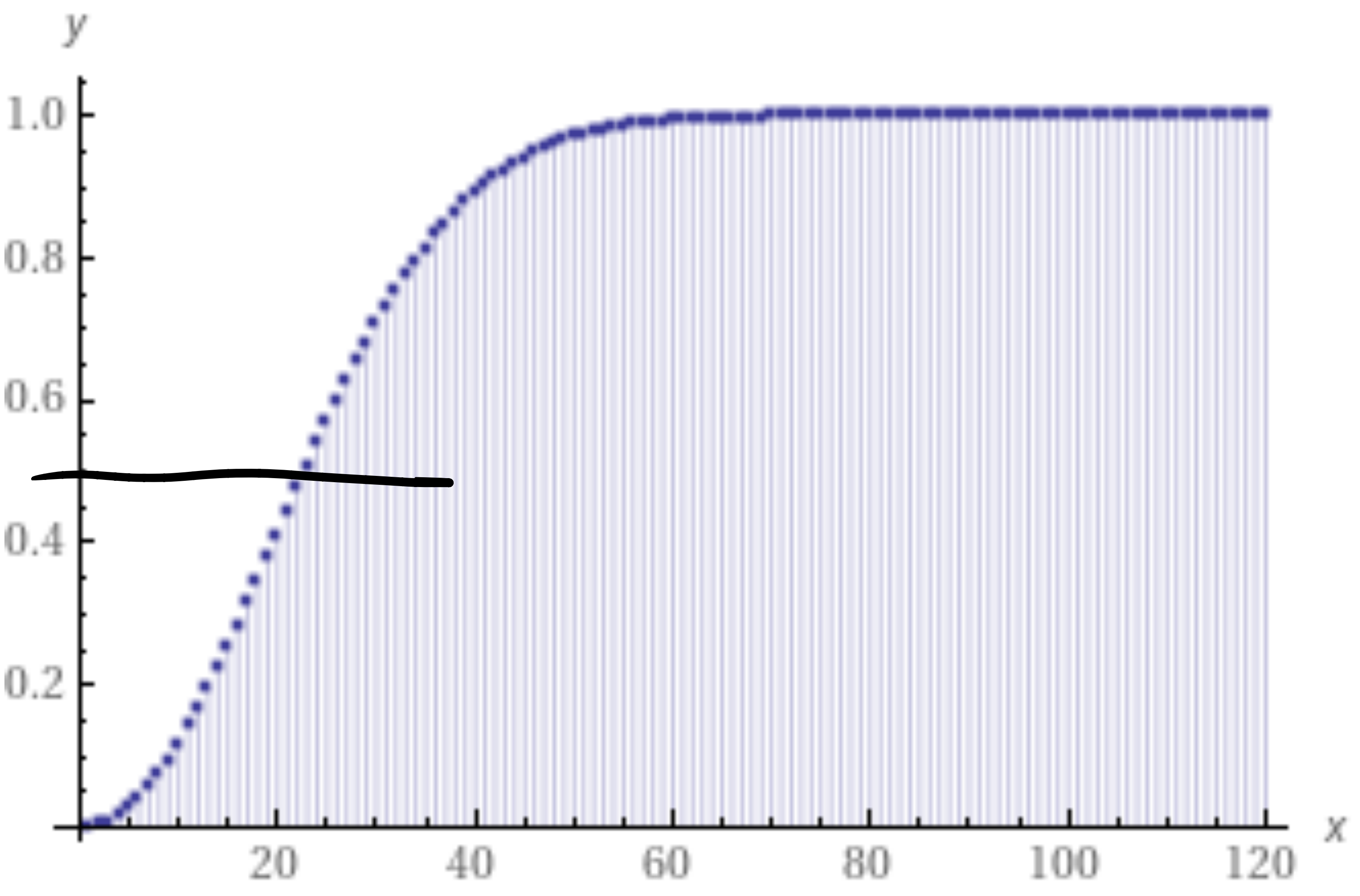
$\mathbb{P}(at\ least\ 2\ people\ in\ the\ room\ share\ a\ birthday)$

$1\ -\mathbb{P}(no\ one\ in\ the\ room\ shares\ a\ birthday)$ (complementation)

$1-\frac{365!}{(365-n)!\cdot 365^n}$

$\frac{|E|}{|\Omega|}$

Let’s graph that out…



(x from 0 to 120)

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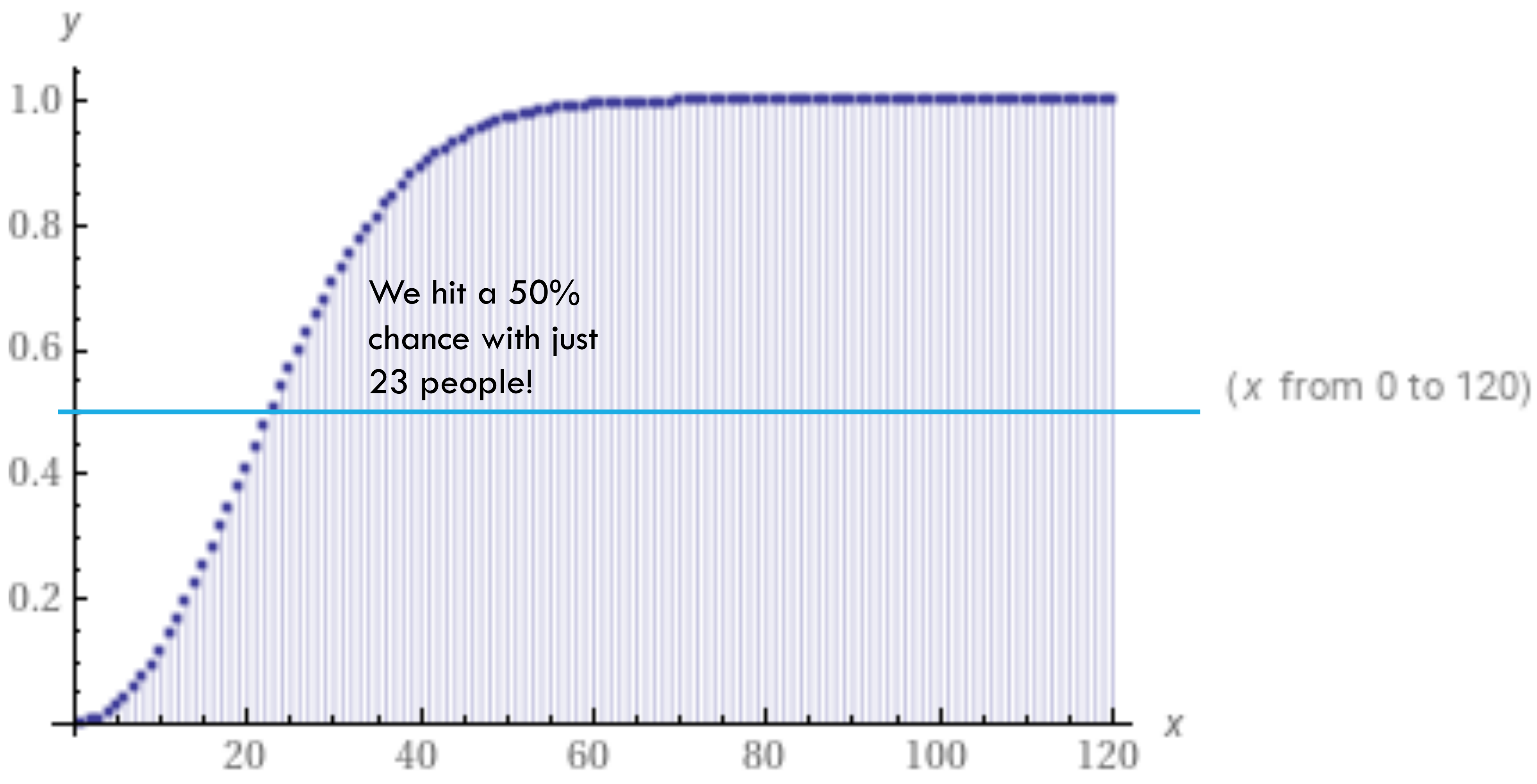
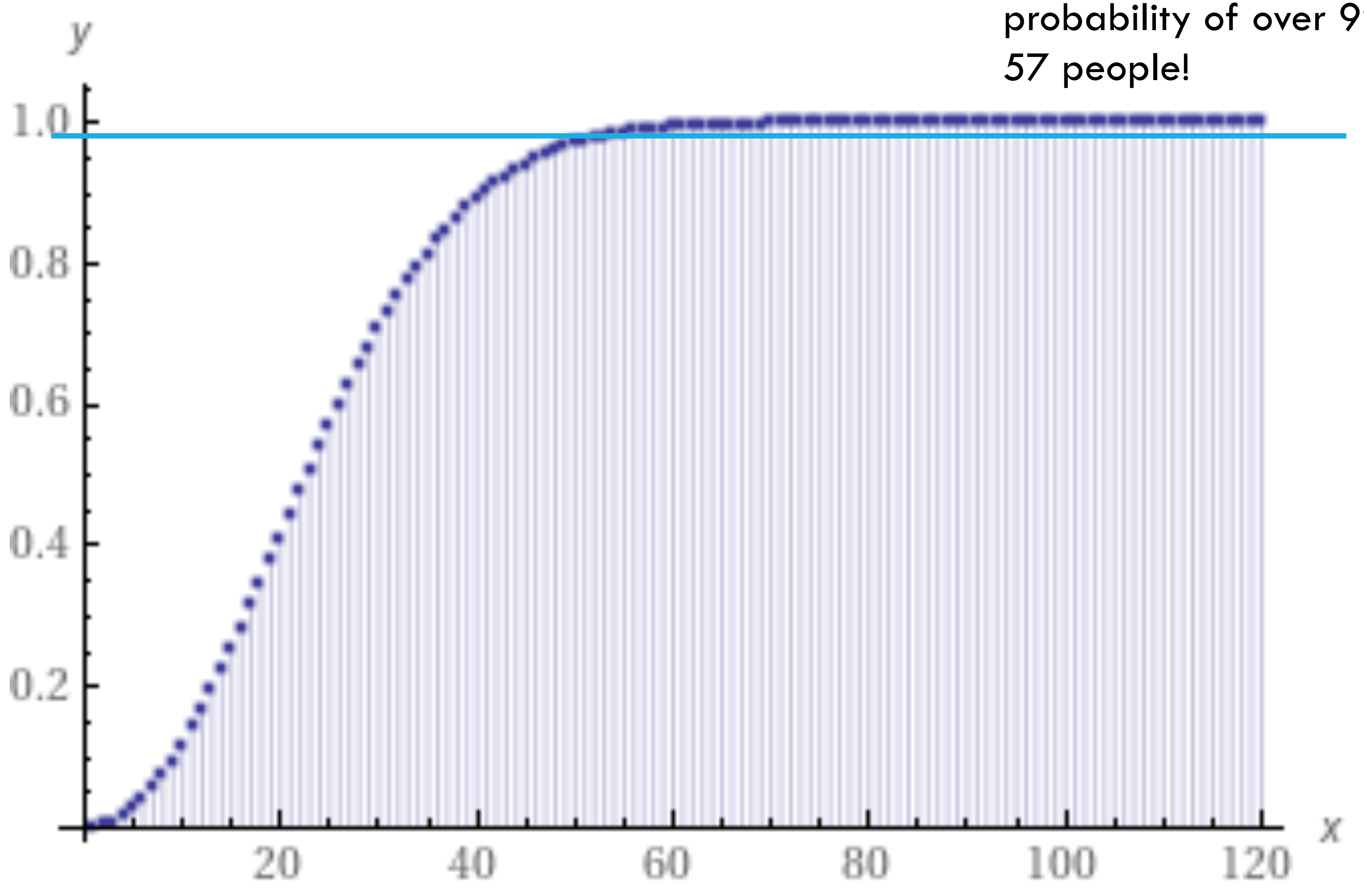


Let's graph that out…

Even more surprisingly, we get a probability of over 99% with just 57 people!



(x from 0 to 120)

Why?

It turns out that human brains find thinking about probabilities difficult!

Our brains are a bit selfish! When it comes to the probability that someone shares our birthday, that would be $\frac{1}{365}$ - not quite so likely. But we were looking at any pair's birthday.

In a group of n, we have $\binom{n}{2}$ pairs of people, which grows quadratically with n. So $\mathbb{P}(E) \to 1$ very quickly ☺ But our brains prefer to think about probabilities linearly rather than exponentially… so here we are.

More Examples

Dice Rolls

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see at least one 3 in the two rolls.

$$\mathbb{P}(roll \geq one\ 3) = 1 - \mathbb{P}(roll < one\ 3)\ [complementation]$$

$$= 1 - \mathbb{P}(no\ 3s\ rolled)$$

$$= 1 - \frac{|ways\ to\ roll\ no\ 3s\ in\ two\ dice\ rolls|}{|\Omega|}$$

$$= 1 - \frac{5 \cdot 5}{6 \cdot 6} = 1 - \frac{25}{36} = \frac{11}{36}$$