No More Counting Please!

CSE 312 Su23
Lecture 3

Outline

So Far

Two more (fun!) techniques

Thinking about counting

A question for you…

Are there at least two people in Seattle with the same number of hairs on their head?

A. Yes

B. No

C. How would I know?

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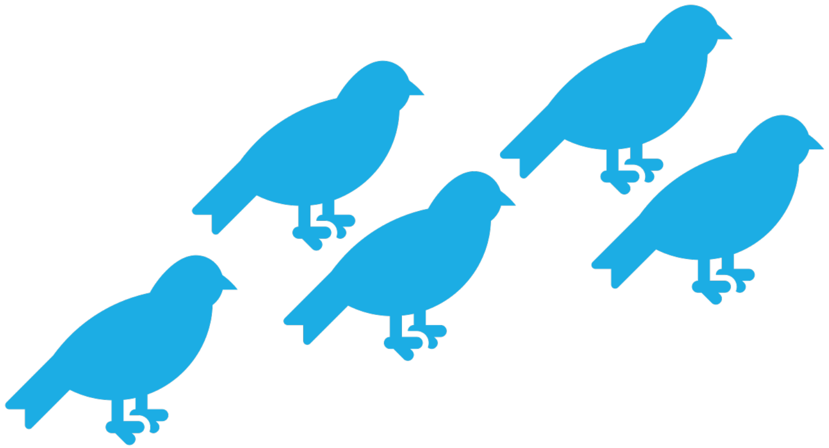
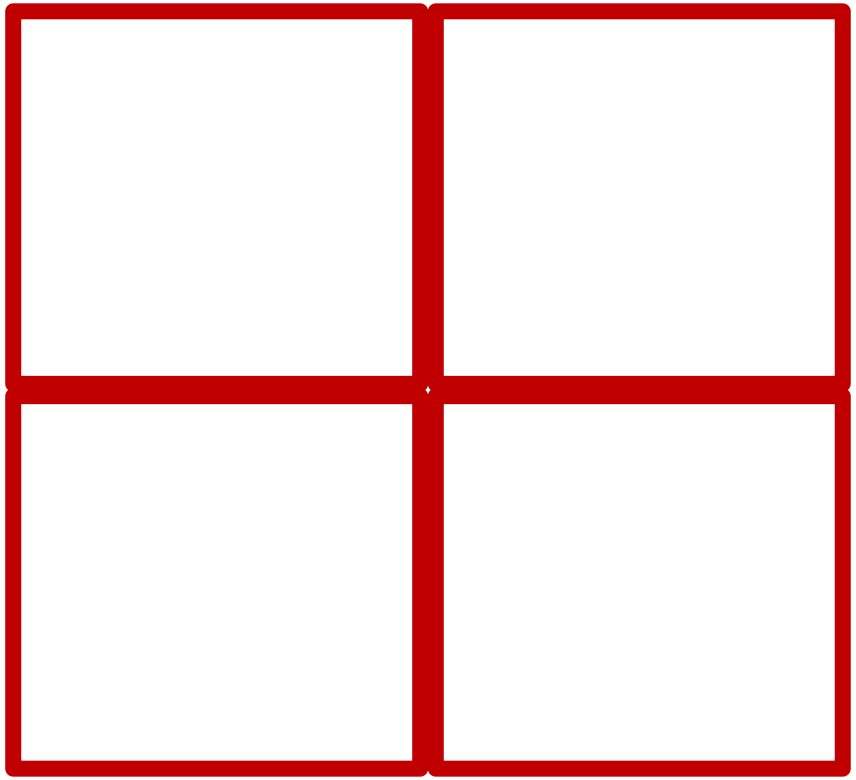
Pigeonhole Principle

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…





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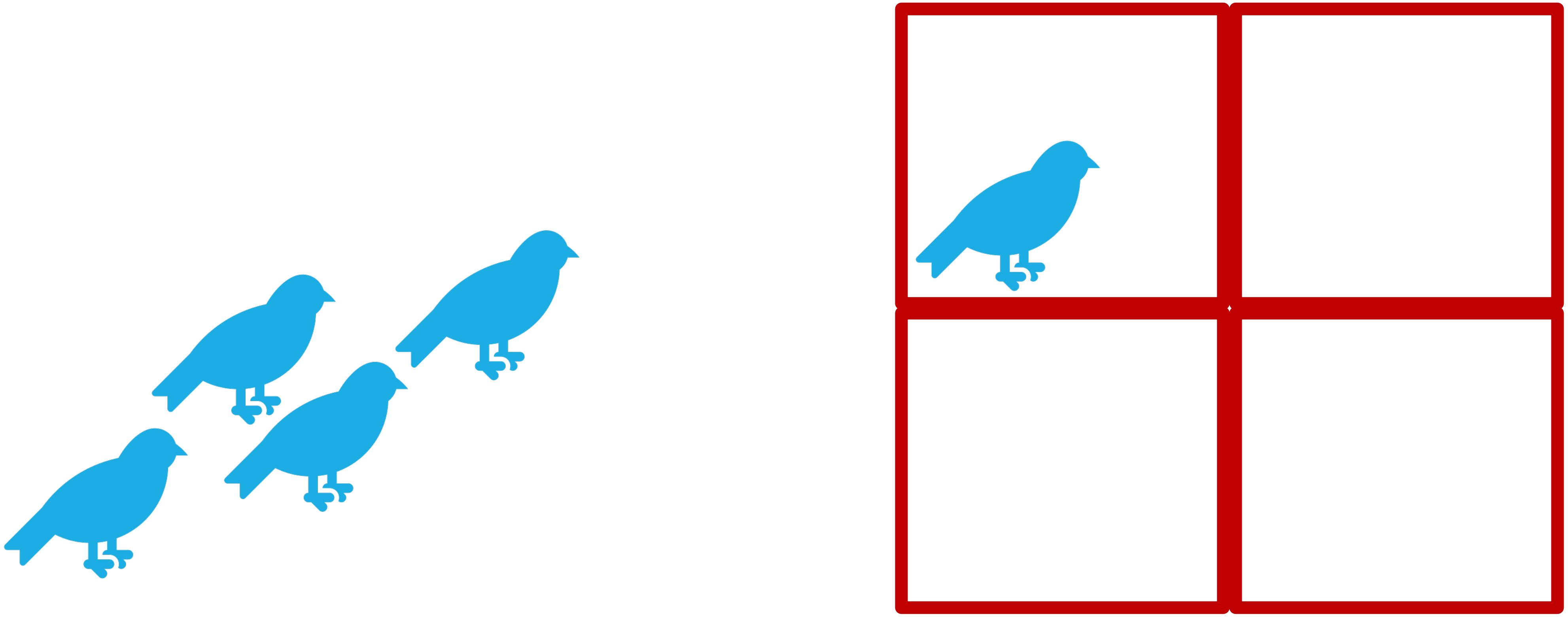

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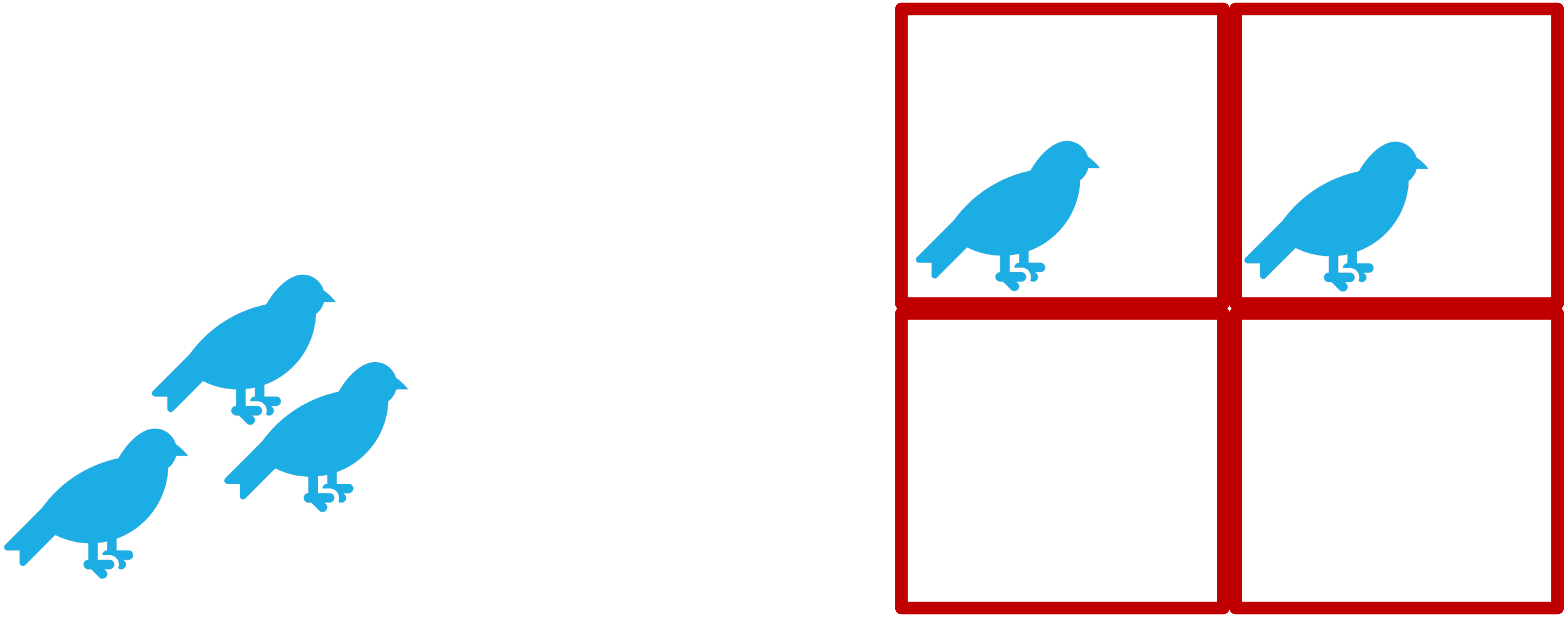

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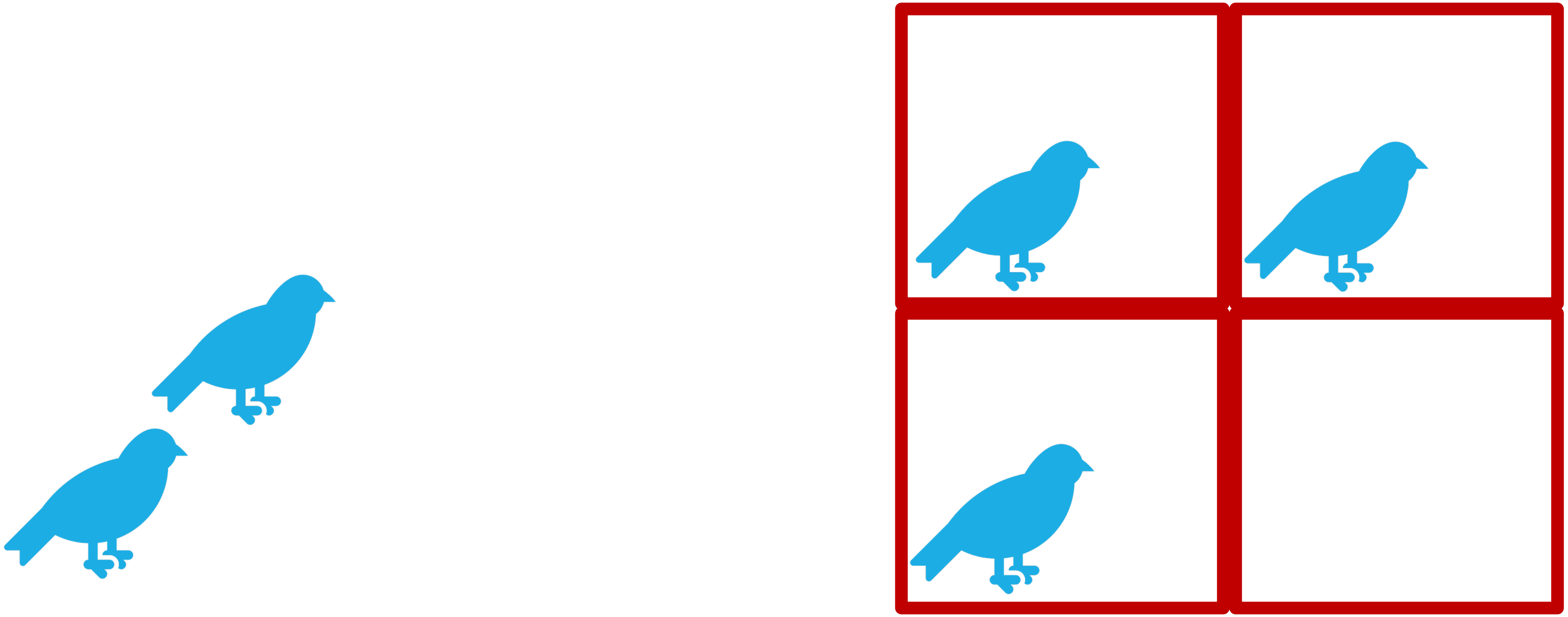

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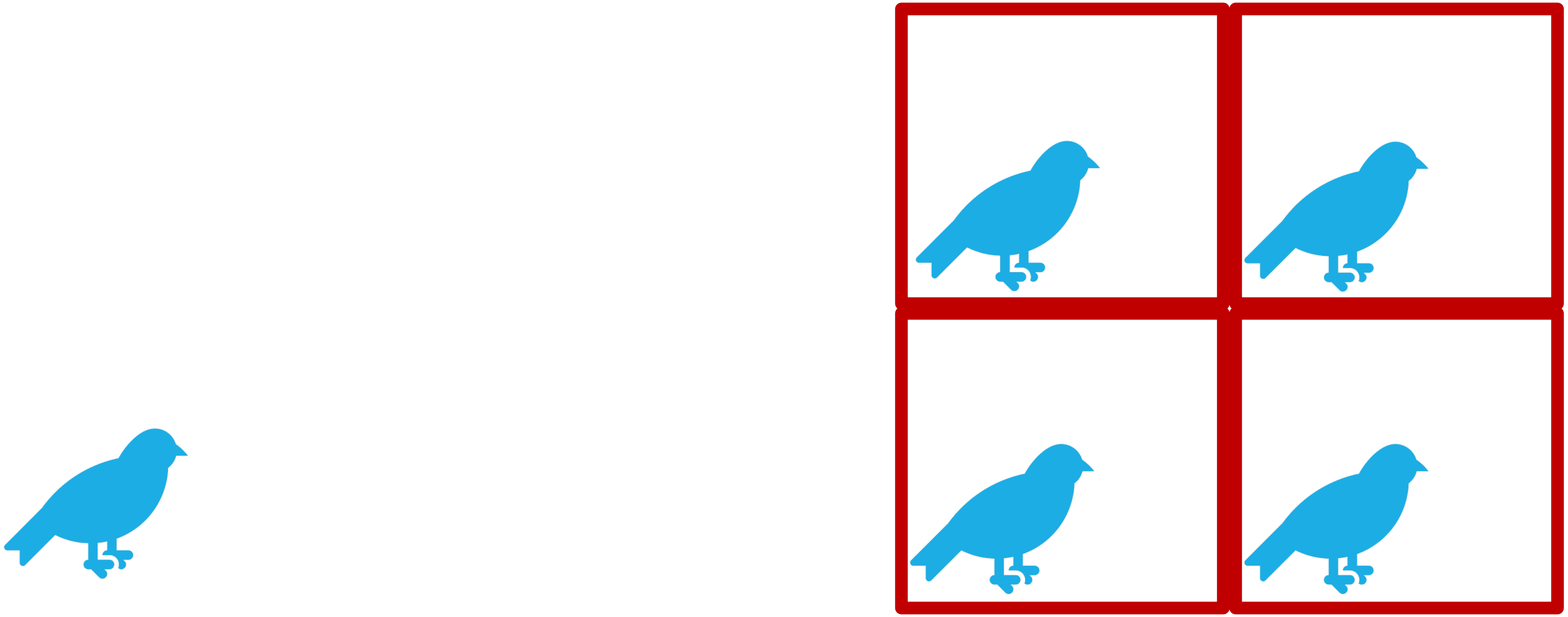
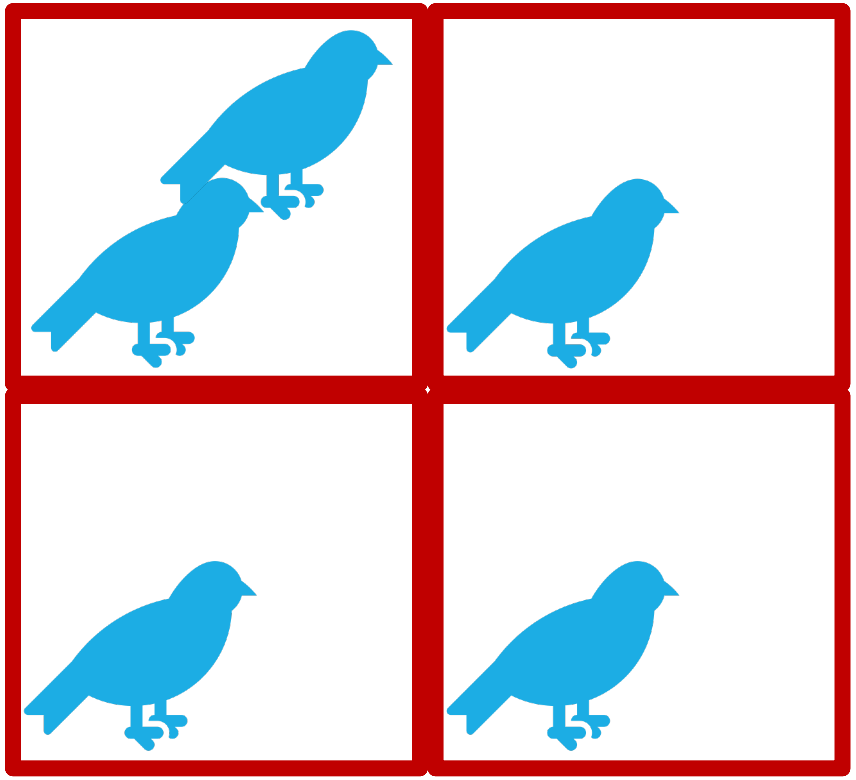


Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.



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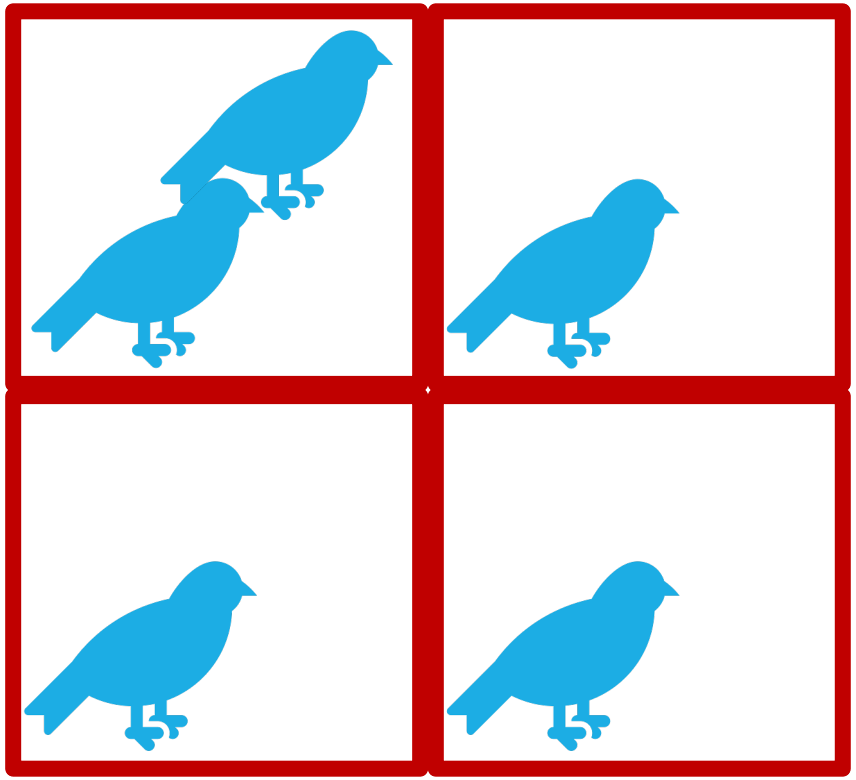


Version 1:
If you have n pigeons and k pigeonholes,
where n > k,
then there is **at least one** pigeonhole
that has at least 2 pigeons.

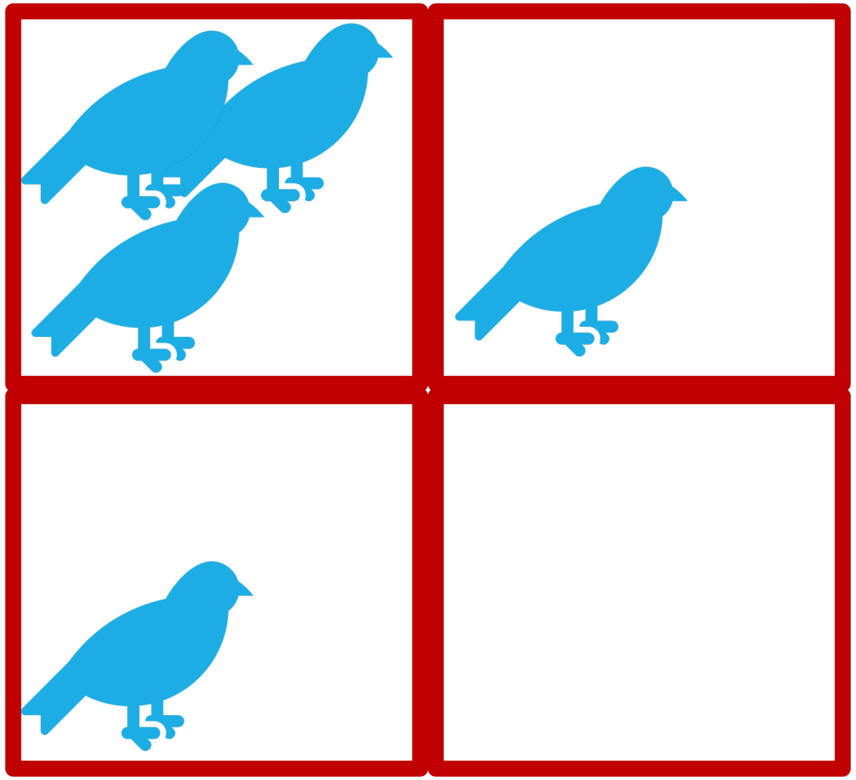
Image credit: Amy Liu

Pigeonhole Principle

If you have 5 pigeons, and place them into 4 holes, then…

At least two pigeons are in the same hole.

It might be more than two.



Strong Pigeonhole Principle

$\left\lceil \frac{n}{k} \right\rceil$

If you have n pigeons, and place them into k holes, then…

If we try to distribute pigeons as evenly as possible, we would first place $\frac{n}{k}$ pigeons in every pigeonhole.

If there are any pigeons left-over, not every pigeonhole will get a pigeon, but at least one hole will get at least one more pigeon.

$\frac{5}{4}$ $\frac{n}{k}$

H1 | | | H4

Then….

$$n \% k = \begin{cases} 0 \\ >0 \Rightarrow 1 \end{cases}$$

Strong Pigeonhole Principle

5 pigeons

4 pigeonholes

If you have n pigeons and k pigeonholes, then there is **at least one** pigeonhole that has at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons.

$\lceil a \rceil$ is the "ceiling" of a (it means always round up, $\lceil 1.1 \rceil = 2$, $\lceil 1 \rceil = 1$).

Be careful: we **aren't** making a guarantee about **every** pigeonhole.

$\frac{n}{k}$

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: Classes

Pigeonholes: Quarters

Mapping: which classes in a quarter

$$\geq \left\lceil \frac{n}{k} \right\rceil \geq \left\lceil \frac{10}{3} \right\rceil = 4$$

An example

If you have to take 10 classes, and have 3 quarters to take them in, then…

Pigeons: The classes to take

Pigeonholes: The quarter

Mapping: Which class you take the quarter in.

Applying the (generalized) pigeonhole principle, there is at least one quarter where you take at least $\left\lceil \frac{10}{3} \right\rceil = 4$ courses.

What's the point?

Well, for one… we can prove some fun things.

Are there at least two people in Seattle with the same number of hairs on their head?

pigeons

Pigeonholes

700000

150000

$$\left\lceil \frac{700000}{150000} \right\rceil = 5$$

What’s the point?

Well, for one… we can prove some fun things.

Are there at least two people in Seattle with the same number of hairs on their head?

Humans are said to have up to 150,000 hairs on their head.

There are over 700,000 people living in Seattle.

So, there must be at least one ‘hair count’ that at least $\left\lceil \frac{700{,}000}{150{,}000} \right\rceil = 5$ people have!

No, what's the point for real?

Being for real….

- In CS, we can show some problems can't be solved. For e.g., a universal lossless compression algorithm is impossible
- It is also the basis of proving that a language is not regular, as you may remember from 311
- Can be used to derive Ω lower bounds on some algorithms
- A LOT of graph theory stuff

Infinite Strings

finite states

Practical Tips

When the pigeonhole principle is the right tool, it's usually the first thing you'd think of or the absolute last thing you'd think of.

For **really** tricky ones, we'll warn you in advance that it's the right method

When applying the principle, say:

- What are the pigeons
- What are the pigeonholes
- How do you map from pigeons to pigeonholes

Look for – a set you're trying to divide into groups, where collisions would help you somehow.

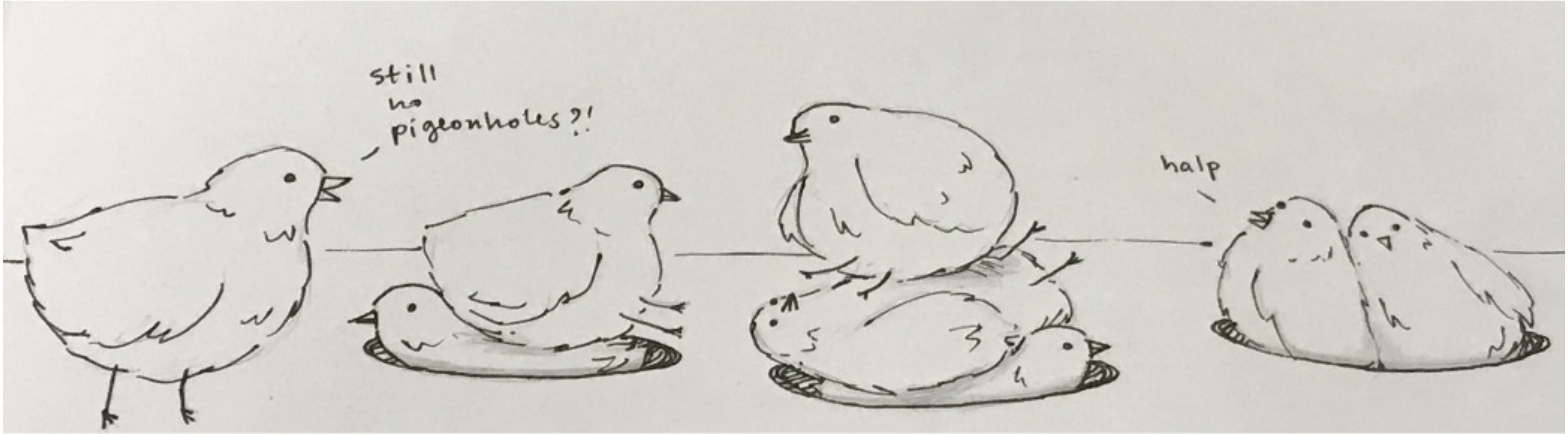


Image credit: Amy Liu

Stars and Bars

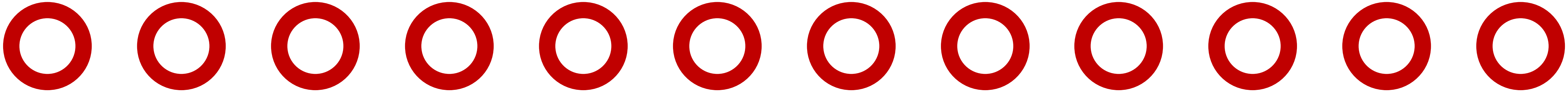
One More Counting Rule

You're going to buy 12 donuts.

There are chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn't matter)

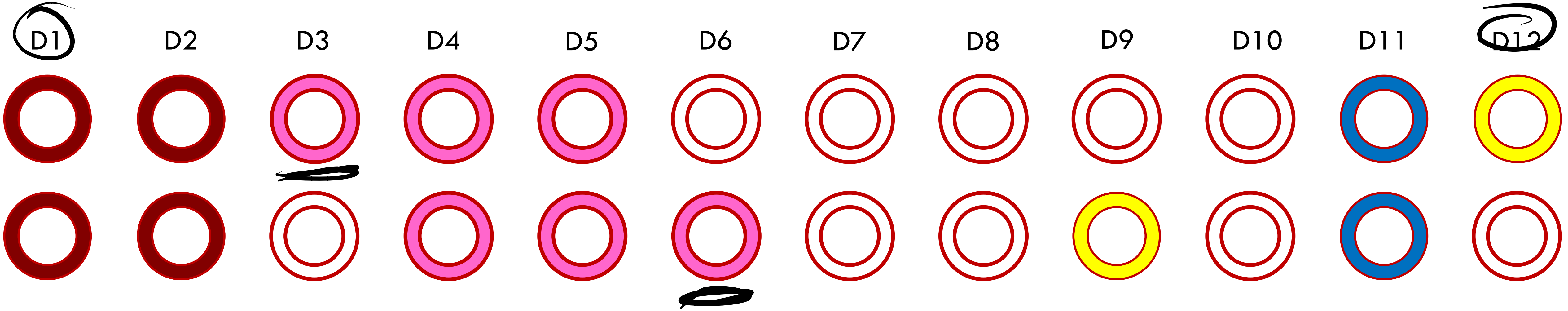


Indistinguishable donuts?

Buy 12 donuts, of types chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types)

How many different donut boxes can you buy?

Consider two boxes the same if they contain the same number of every kind of donut (order doesn't matter) -> <u>which donut is a particular flavor doesn't matter</u>

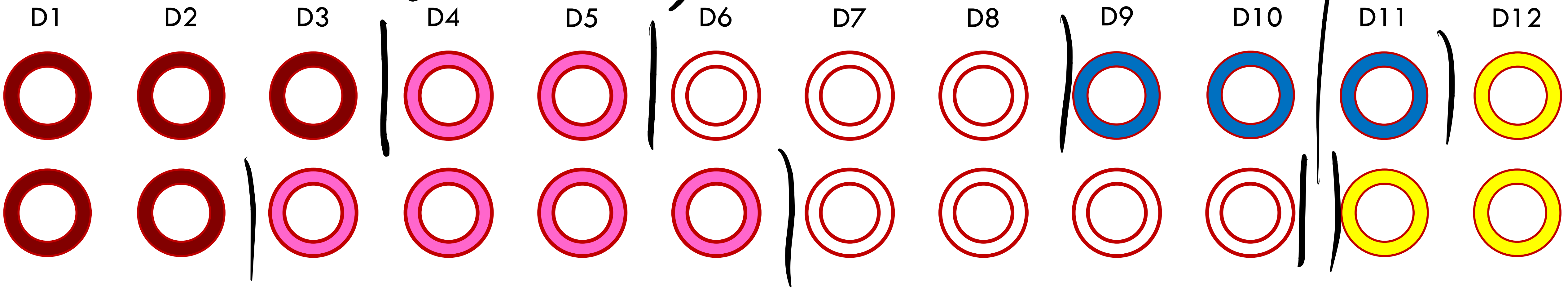


Idea

Buy 12 donuts, of types chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types). How many different donut boxes can you buy?

1. Order by flavor
2. Distribution
3. Arrangement

dividers



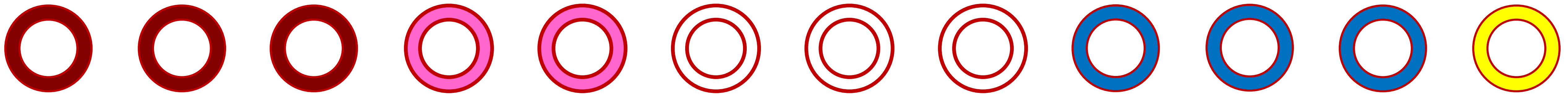
Idea

Buy 12 donuts, of types chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types). How many different donut boxes can you buy?

- Order the donuts, by type.

- We are interested in how many ‘donuts/type’ distributions we can have

- To do that, we could look at the ‘boundaries’ between flavors. Perhaps, by placing ‘dividers’ at the boundaries. Since a type could have 0 donuts, two dividers could be right next to each other.



Idea

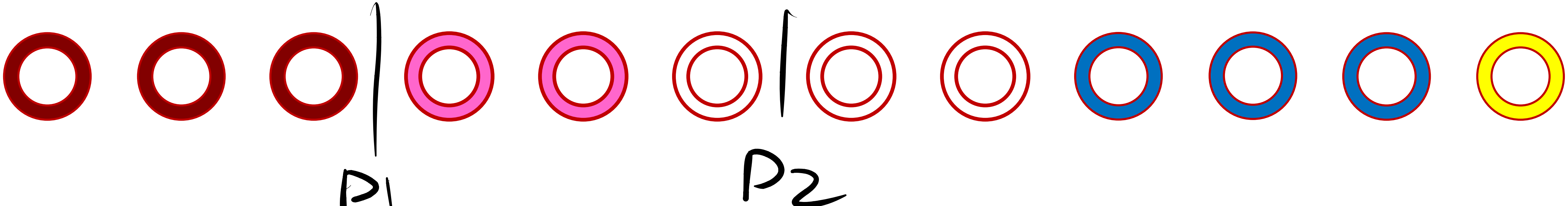
$\binom{16}{4}$

Buy 12 donuts, of types chocolate, strawberry, coconut, blueberry, and lemon (i.e. 5 types). How many different donut boxes can you buy?

- So, our problem has been reduced to: how many ways are there to place dividers amongst the donuts, or how many ways are there to arrange the donuts and dividers?

\# of dividers: $5-1=4$

\# of potential places - $12+4$



Ordering Donuts Into Flavors

With 5 distinct flavors, how many dividers are we placing? 4 dividers.

$\binom{16}{4}$

12 donuts + 4 dividers

Ordering Donuts Into Flavors

With 5 distinct flavors, how many dividers are we placing? 4 dividers.

With 12 donuts and 4 bars, we are arranging 16 objects in a line.

How many ways to do this?

There are 16 possible positions and 4 bars, so there are $\binom{16}{4} = \binom{12+5-1}{5-1}$ ways to arrange the donuts and bars.

Ordering Donuts Into Flavors

There are 16 possible positions and 4 bars, so there are $\binom{16}{4} = \binom{12+5-1}{5-1}$ ways to arrange the donuts and bars.

Note that this is another formulation of the "ways to arrange a string" question, where all the donuts are the same 'letter' and all the bars are the same 'letter': i.e., we are arranging a string like 'ddddddddddddbbbb', in $\frac{16!}{4!12!}$ ways.

In General

To pick n objects from k groups (where order doesn't matter and every element of each group is indistinguishable), use the formula:

$$\binom{n+k-1}{k-1}$$

The counting technique we did is often called "stars and bars"

using a "star" instead of a donut shape, and calling the dividers "bars"

We’ve seen lots of ways to count

Sum rule (split into disjoint sets)

Product rule (use a sequential process)

Combinations (order doesn’t matter)

Permutations (order does matter)

Principle of Inclusion-Exclusion (intersecting sets)

Complementary Counting (when the complement is easier to count)

“Stars and Bars” $\binom{n+k-1}{k-1}$ (indistinguishable objects into distinguishable bins)

Niche Rules (useful in very specific circumstances)

Pigeonhole Principle

Thinking about Counting

Practice

How do we know which rule to apply?

With practice you can pick out patterns for which ones might be plausible: think about the **key properties** of these rules and learn to **identify those same properties** in problems

But if as you're working you realize things are getting out of control, put it aside and try something different.

Cards

order matters ← $\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$ 2

A "standard" deck of cards has 52 cards. Each card has a suit
diamonds ♦,
hearts ♥,
clubs ♧,
spades ♠

and a value (Ace,2,3,4,5,6,7,8,9,10,Jack,Queen, King).

A "5-card-hand" is a set of 5 cards

How many five-card "flushes" are there? – a flush is a hand of cards all of the same suit.

$\binom{4}{1}\binom{13}{5}$ 1

Cards

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Cards

How many five-card “flushes” are there? – a flush is a hand of cards all of the same suit.

Way 1: How can I describe a flush? Which suit it is, and which values:

$$\binom{4}{1} \cdot \binom{13}{5}$$

Cards

Way 2: Pretend order matters. The first card can be anything,

After that, you'll have 12 options (the remaining cards of the suit), then 11, …

Then divide by $5!$, since order isn't supposed to matter.

$$\frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5!}$$

This is the same number as what we got on the last slide!

A Solution with a Problem

You wish to count the number of 5-card hands with at least 3 aces.

There are 4 Aces (and 48 non aces)

$$\underbrace{\binom{4}{3}}_{}\cdot\underbrace{\binom{49}{2}}_{}$$

Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

What's wrong with this calculation?

What's the right answer?

Fill out the poll everywhere so Shreya knows how long to explain
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A Solution with a Problem

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$\binom{4}{3} \cdot \binom{49}{2}$ - Choose the three aces. Then of the 49 remaining cards (the last ace is allowed as well, because we're allowed to have all 4)

If I draw A♦, which term in my calculation drew that card?

A Solution with a Problem

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If I draw A♦, which term in my calculation drew that card?

There's no way for us to know:

{A♧, A♠, A♦} {A♥, K ♠} And {A♧, A♠, A♥}, {A♦, K ♠}

Are two different choices of the process, but they lead to the same hand!

Sleuth's Criterion

For each object constructed, there should be **exactly one set of choices** in the sequential process that gets us there.

Otherwise…

If there is **no sequence** of choices that could construct an object in the set we are counting, we have **undercounted**.

If there are **many sequences** of choices that could construct an object in the set we are counting, we have **overcounted.**

Sleuth's Criterion in Context

For each **5-card hand with at least 3 aces** constructed, there should be exactly one set of choices in the sequential process that gets us there.

Otherwise…

If there is **no sequence** of choices that could construct **a 5-card hand with at least 3 aces**, we have **undercounted**.

If there are **many sequences** of choices that could construct **a 5-card hand with at least 3 aces**, we have **overcounted.**

Fixing The Overcounting

A Solution with a Problem

One solution is to try again…

The problem was trying to account for the “at least” – look at the disjoint sets and count them separately.

A 5-card hand with at least 3 aces could have (1) exactly 3 aces or (2) exactly 4 aces.

Disjoint $\rightarrow$ sum rule

(1) $\binom{4}{3}\binom{48}{2}$ $+$ (2) $\binom{4}{4}\binom{48}{1}$

A Solution with a Problem

One solution is to try again…

The problem was trying to account for the “at least” – look at the disjoint sets and count them separately.

A 5-card hand with at least 3 aces could have (1) exactly 3 aces or (2) exactly 4 aces.

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

If there are exactly 3 aces, we choose which 3 of the 4, then choose which 2 cards among the 48 non-aces. If all 4 aces appear, then one of the remaining 48 cards finishes the hand. Applying the sum rule completes the calculation.

Takeaway

It's hard to count sets where one of the conditions is "at least X"

You usually need to break those conditions up into disjoint sets and use the sum rule.

A Solution with a Problem

Another solution is to count exactly which hands appear more than once, and how many times each appears and compensate for it.

$\{$ hands w/ 4 Aces $\}$

$$\binom{4}{3}\binom{49}{2} - 3 \times \underbrace{\left(\#\text{ of hands w/ 4 A}\right)}_{\binom{4}{4}\binom{48}{1} = 48}$$

When do we overcount?

If there are exactly 4 Aces in the hand, then we count the hand 4 different times (once for each ace as an "extra" one):

{A♣, A♠, A♦}, {A♥, ?}

{A♣, A♠, A♥}, {A♦, ?}

{A♣, A♥, A♦}, {A♠, ?}

{A♥, A♠, A♦}, {A♣, ?}

How much do we overcount?

How many hands are we counting 4 times?

There are 48 such hands (one for every card – all non-aces – that could be “?” on the last slide)

So we’ve counted $3 \cdot 48$ processes that shouldn’t count.

That would give a corrected total of $\binom{4}{3} \cdot \binom{49}{2} - 3 \cdot 48$

This is the same number as we got our other method.