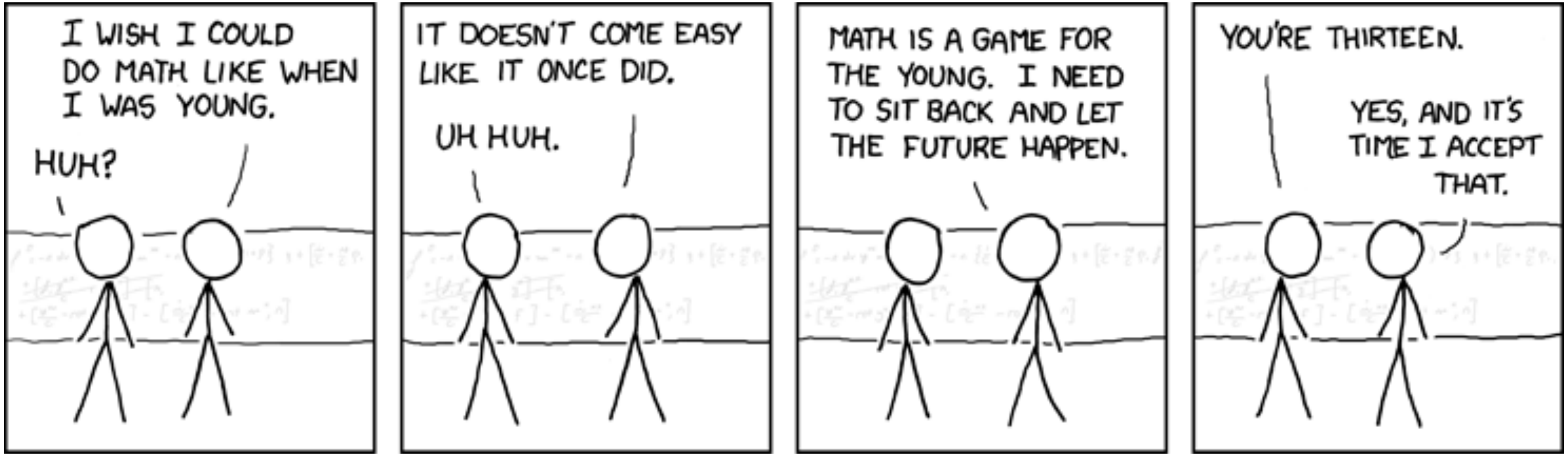

https://xkcd.com/447/

More Counting

CSE 312 Su23
Lecture 2

Announcements

- HW1 is out - all the material you need for it will be covered by the end of this lecture
- Don't forget to typeset - there are some helpful links on typesetting on the course website
- Justify your answers! We want to see your thought process
- Coding question: we have an environment on Ed for convenient set up + running tests, but final submission must be done to the Gradescope autograder

Where Are We?

Last time:

Sum and Product Rules

Complementary counting

Permutations, begin

Today:

Combinations and Permutations (even fancier counting!)

Combinatorial Proofs (or proof by double counting!)

Recap: Permutations

How many length 3 sequences are there consisting of distinct elements of $\{1,2,3\}$.

Step 1: 3 options for the first element.

Step 2: 2 (remaining) options for the second element.

Step 3: 1 (remaining) option for the third element.

$3 \cdot 2 \cdot 1$.

Recap: Permutations

That formula shows up a lot.

The number of ways to “permute” (i.e., “order n distinct elements”) is “n factorial”

n factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 1$$

We only define $n!$ for natural numbers n.

As a convention, we define: $0! = 1$.

Distinct Letters

How many length 5 strings are there over the alphabet $\{a, b, \ldots, z\}$ where each string does not repeat a letter.

E.g. "azure" is an allowed string, but "steve" is not, nor is "abcdef"

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$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$$

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$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 = \frac{26 \cdot 25 \cdot \ldots \cdot 22 \cdot 21 \cdot 20 \cdot \ldots \cdot 2 \cdot 1}{21 \cdot 20 \cdot \ldots \cdot 2 \cdot 1}$$

$$= \frac{26!}{21!}$$

In General

k-permutation

The number of k-element sequences of distinct symbols from a universe of n symbols is:

$$\boldsymbol{P(n,k) = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}}$$

Said out loud as "P n k" or "n permute k" or "n pick k"

Alternative notation: ${}_nP_k$

Edge cases: $P(n,n) = n!$, $P(n,0) = 1$, $P(n,k)$ for $k < 0$ or $k > n$ is undefined.

Change it slightly

How many **subsets** of size 5 are there of $\{a, b, \ldots, z\}$

Subset properties:
- As before, we have sets of ‘distinct’ elements
- But for subsets, order doesn’t matter

That means…

$\{a, z, u, r, e\}$ is the same set as $\{a, u, r, e, z\}$

(even though “azure” and “aurez” are different strings)

Clever approach – count two ways

Let’s go back to looking at the ways to have an ordered list for a minute.

We know from earlier that this is a permutation: ordering 5 elements out of a universe of 26, or $P(26,5)$.

Is there another way to create an ordered list… possibly using a chosen subset?

Clever approach – count two ways

Let's go back to looking at the ways to have an ordered list for a minute.

We know from earlier that this is a permutation: ordering 5 elements out of a universe of 26, or $P(26{,}5)$.

Is there another way to create an ordered list… possibly using a chosen subset?

How about this sequential process:

Step 1: Choose a subset (of 5 elements, from 26)

Step 2: Put the subset (of 5 elements) in order

These two methods are counting the **same thing**, so they should be equal:

$$\frac{26!}{(26-5)!} = X \cdot 5!$$

Clever approach – count two ways

$$\frac{26!}{(26-5)!} = X \cdot 5!$$

$$\frac{26!}{(26-5)! \cdot 5!} = X$$

So the number of size-5 subsets of a size-26 set is:

$$\frac{26!}{(26-5)!\,5!}$$

Number of Subsets

k-combination

The number of k-element subsets from a set of n symbols is:

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!\,(n-k)!}$$

Said out loud "n choose k" (or sometimes: "n combination k")

Lots of notation:

$_nC_k$ or $\binom{n}{k}$ or $C(n,k)$ all mean "number of size-k subsets of a size-n set."

Common case: $\binom{n}{1} = n$

Edge cases: $\binom{n}{0} = 1, \binom{n}{n} = 1$; $\binom{n}{k}$ for $k < 0$ or $k > n$ is undefined.

Second Takeaway

The second way of counting hints at a generally useful trick:

Pretend that order does matter, then divide by the number of orderings of the parts where order doesn't matter.

For example, here's another way to get the formula for combinations:

You have n elements. Put them in order, take the first k as your set.

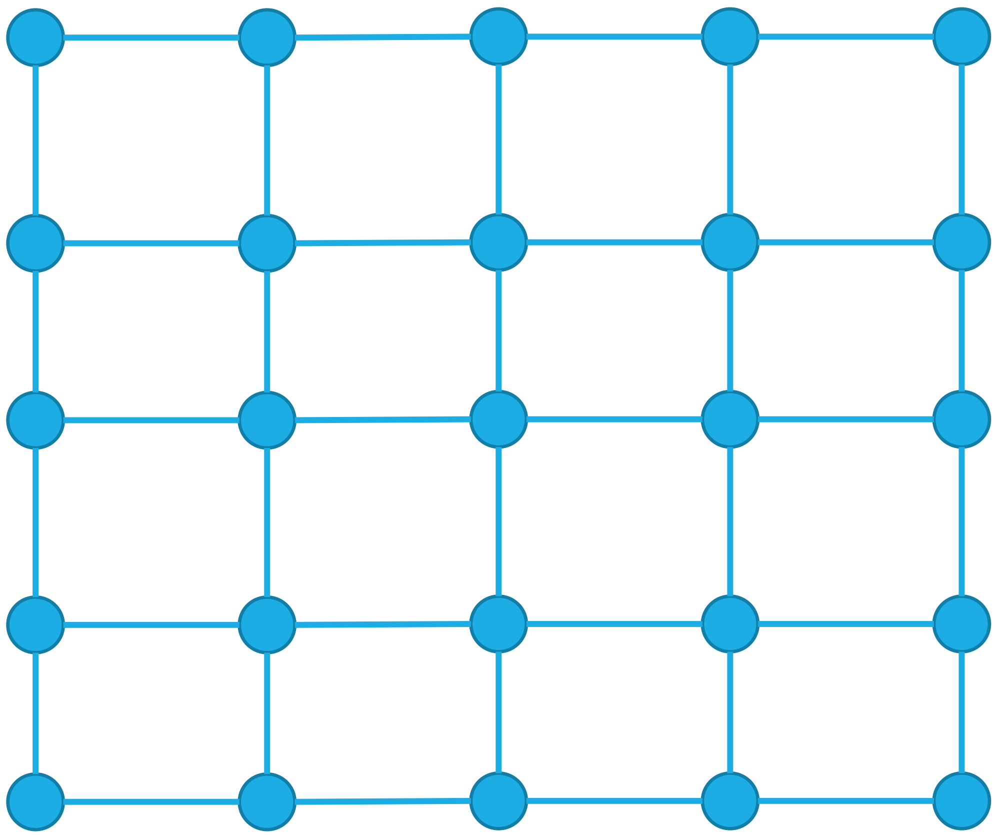
$n!$ Orderings overall. We've overcounted because:

Among the first k, order doesn't matter between them. Divide by $k!$.

Among the last $n - k$, order doesn't matter between them. Divide by $(n-k)!$.

$$\frac{n!}{k!\,(n-k)!}$$

Path Counting



We're in the lower-left corner, and want to get to the upper-right corner.

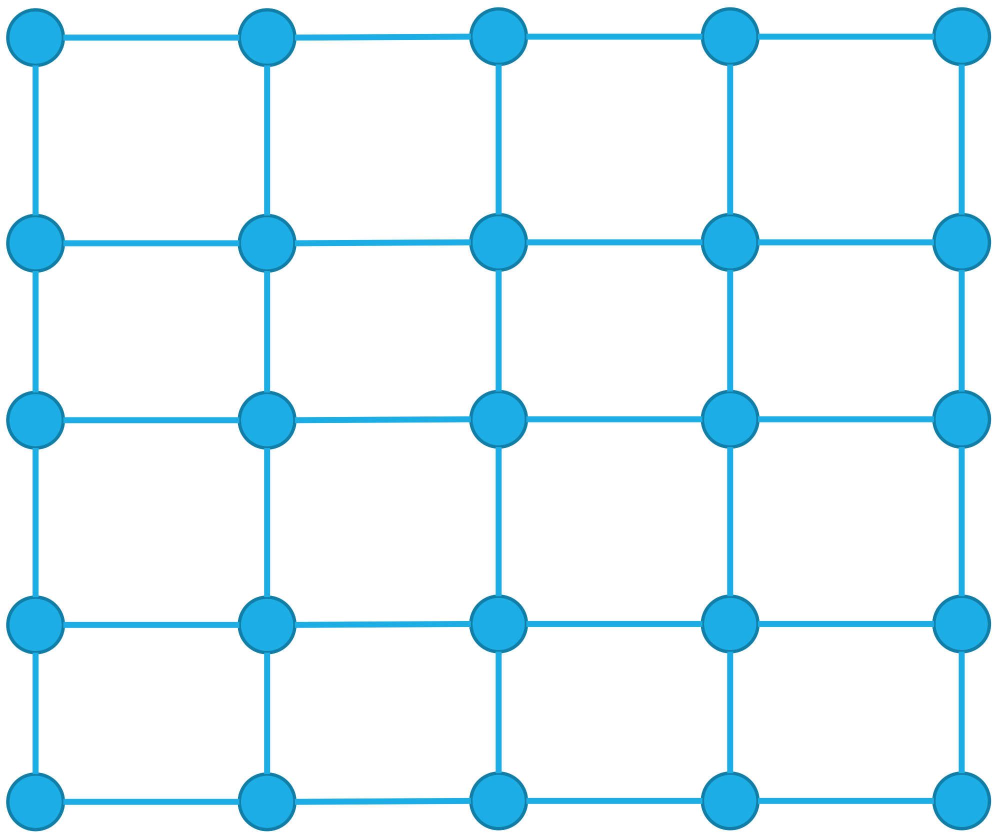
We're only going to go right and up.

How many different paths are there?

$A.\ 2^8$
$B.\ P(8,4)$
$C.\ \binom{8}{4}$
D. Something else

Fill out the poll everywhere so Shreya knows how much to explain
Go to pollev.com/312 and login with your UW identity

Path Counting



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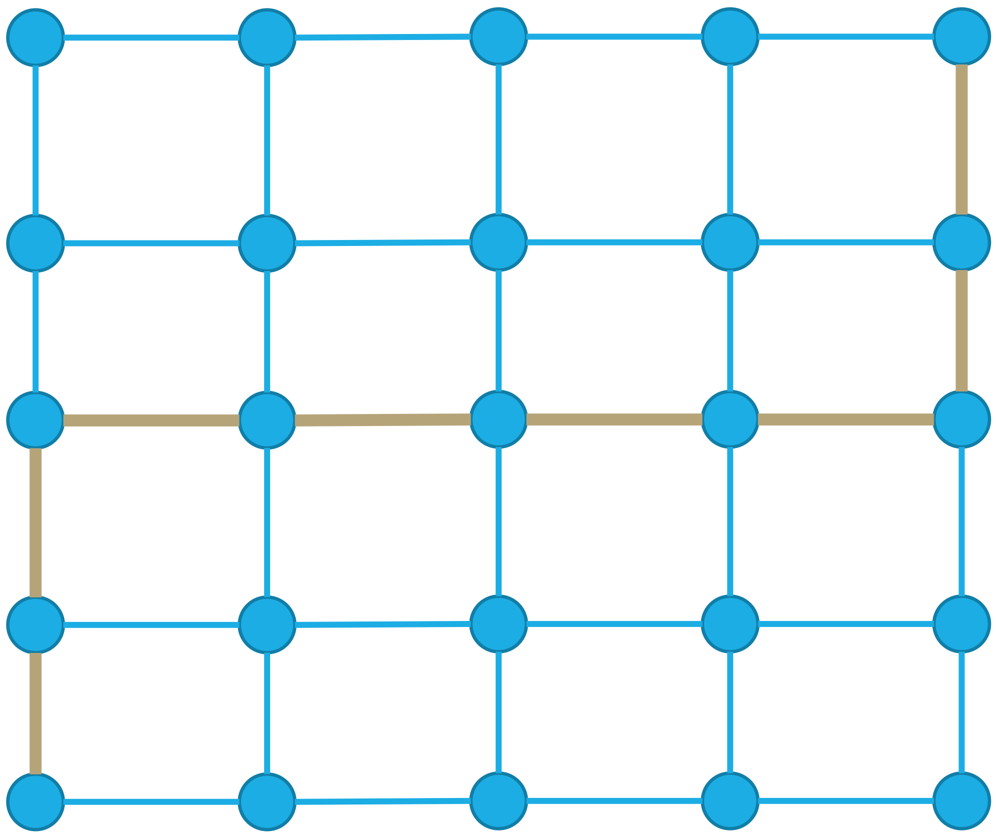
How many different paths are there?

Idea 1:
We're going to take 8 steps. Choose which SET of 4 of the steps will be up (the others will be down).

Path Counting



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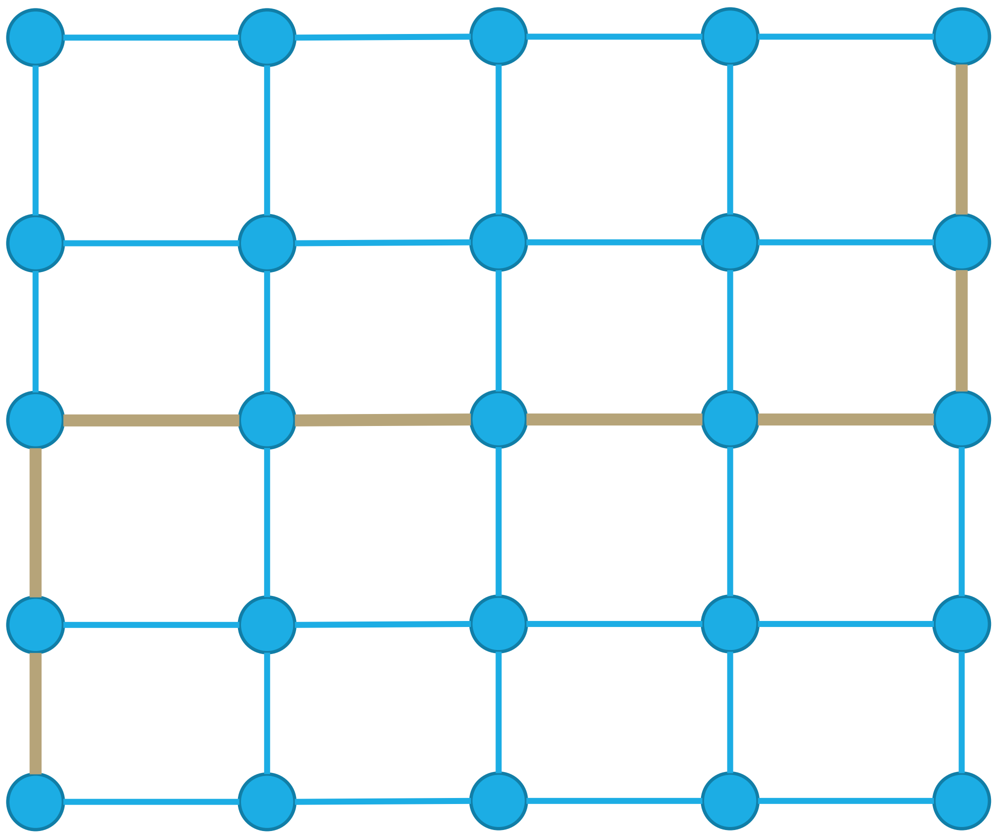
Choose which SET of 4 of the steps will be up (the others will be down).
E.g. $\{1,2,7,8\}$ is
How many size-4 subsets of $\{1,2,3,4,5,6,7,8\}$ are there?

Path Counting



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$\boxed{\binom{8}{4} \text{ is the answer.}}$

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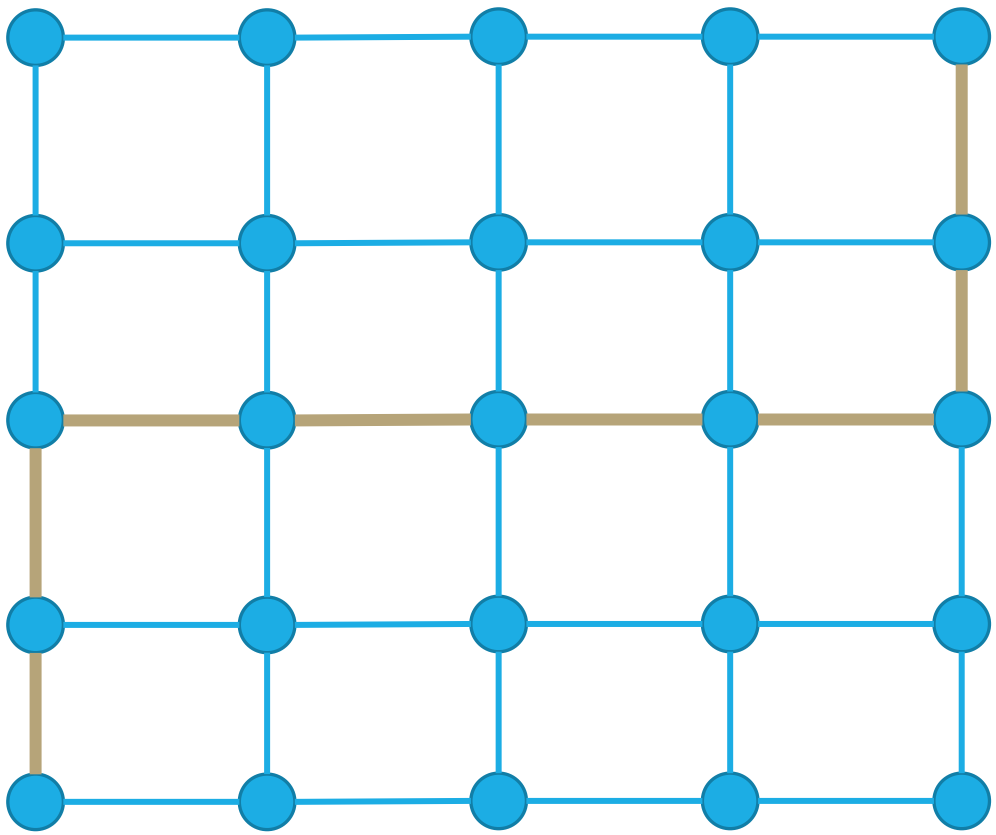

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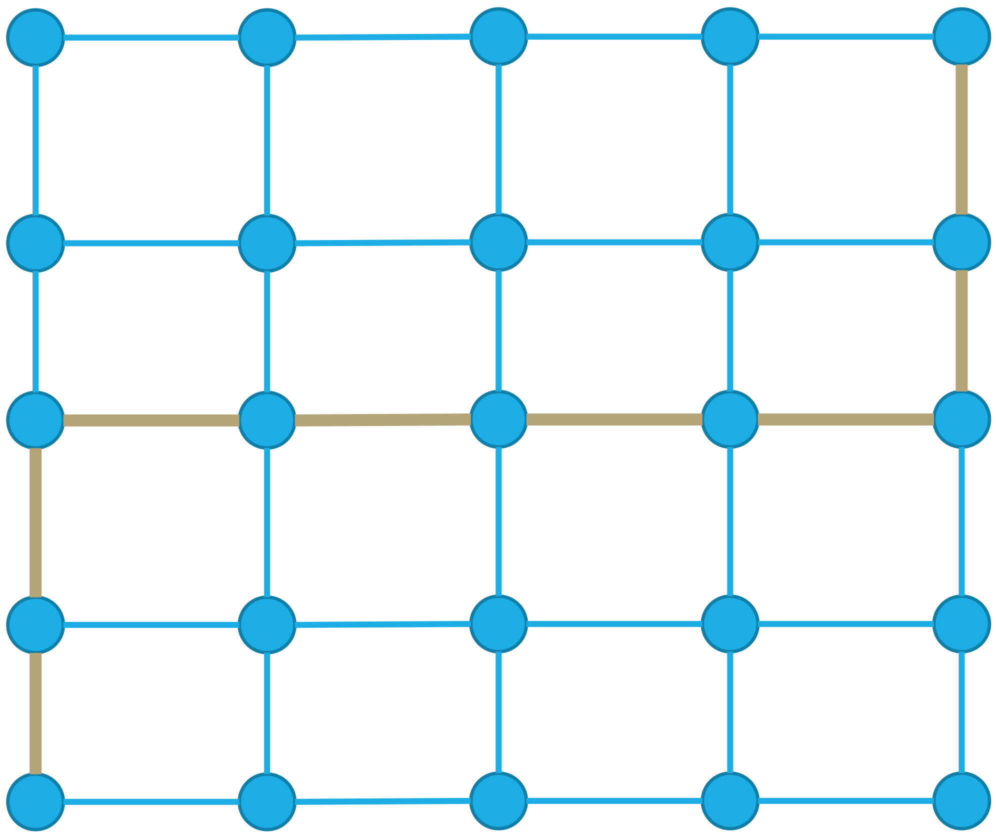
How many different paths are there?

Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \uparrow_C \uparrow_D \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D$ $\quad 8!$

Path Counting



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How many different paths are there?

Idea 2: Introduce artificial ordering

Order $\uparrow_A \uparrow_B \uparrow_C \uparrow_D \rightarrow_A \rightarrow_B \rightarrow_C \rightarrow_D$ $\quad 8!$

Remove the overcounting

Those $4 \uparrow$ are really the same, divide by $4!$

The $4 \rightarrow$ are really the same, divide by $4!$

Total: $\frac{8!}{4! \cdot 4!}$

$\boxed{\binom{8}{4} \text{ is the answer.}}$

Multinomial Coefficients

How many anagrams are there of SEATTLE

(an anagram is a rearrangement of letters).

It's not 7!

That counts SEATTLE and SEATTLE as different things!

Or maybe, SEATTLE and SEATTLE!

Multinomial Coefficients: Approach 1

How many anagrams are there of SEATTLE?

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements SEATTLE? 7!

Multinomial Coefficients: Approach 1

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How have we overcounted? Es relative to each other and Ts relative to each other **2! · 2!**

> Why 2!

Multinomial Coefficients: Approach 1

How many anagrams are there of SEATTLE?

Pretend the order of the Es (and Ts) relative to each other matter (that SEATTLE and SEATTLE are different)

How many arrangements SEATTLE? $7!$

How have we overcounted? Es relative to each other and Ts relative to each other $2! \cdot 2!$

Final answer $\frac{7!}{2! \cdot 2!}$

Multinomial Coefficients: Approach 2

For the letters in SEATTLE,

$$\_\ \_\ \_\ \_\ \_\ \_\ \_$$

First, place the 2 E’s: $\binom{7}{2}$

Then, place the 2 T’s: $\binom{5}{2}$

Then, place the S: $\binom{3}{1}$ and then, place the A: $\binom{2}{1}$, and finally, the L: $\binom{1}{1}$

For a total of: $\binom{7}{2} \cdot \binom{5}{2} \cdot 3 \cdot 2 \cdot 1 = \frac{7!}{2!5!} \cdot \frac{5!}{2!3!} \cdot 3! = \frac{7!}{2!2!}$

Overcounting

How many anagrams are there of GODOGGY?

$\frac{7!}{2!3!}$

One more piece of notation – "multinomial coefficient"

$\binom{7}{2,3}$ is alternate notation for $\frac{7!}{2!3!}$.

In general: $\binom{n}{k_1,k_2,\dots,k_\ell} = \frac{n!}{k_1!\cdot k_2!\cdots k_\ell!}$

Popular notation among mathematicians.

Multinomial coefficients

If we have k types of objects (amongst n total), with n_1 of the first, n_2 of the second, …, n_k of the kth, then the total number of arrangements possible *is*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \ldots n_k!}$$

Combination Facts

Some Facts about combinations

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

First Proof of Symmetry

Proof 1: By algebra

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ Definition of Combination

$= \frac{n!}{(n-k)!k!}$ Algebra (commutativity of multiplication)

$= \binom{n}{n-k}$ Definition of Combination

First Proof of Symmetry

Wasn’t that a great proof.

Airtight. No disputing it.

Got to say “commutativity of multiplication.”

But…do you know *why*? Can you *feel* why it’s true?

Second proof of symmetry

Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

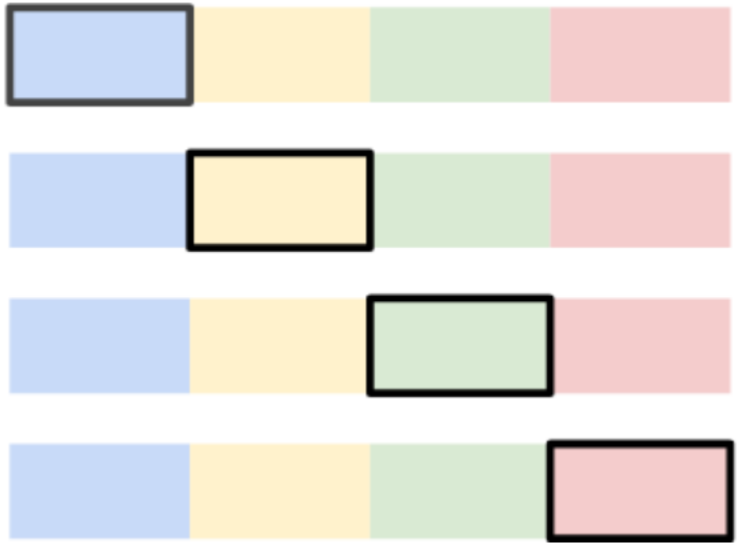
Two **equivalent** ways to choose **k out of n objects** (unordered):

Second proof of symmetry

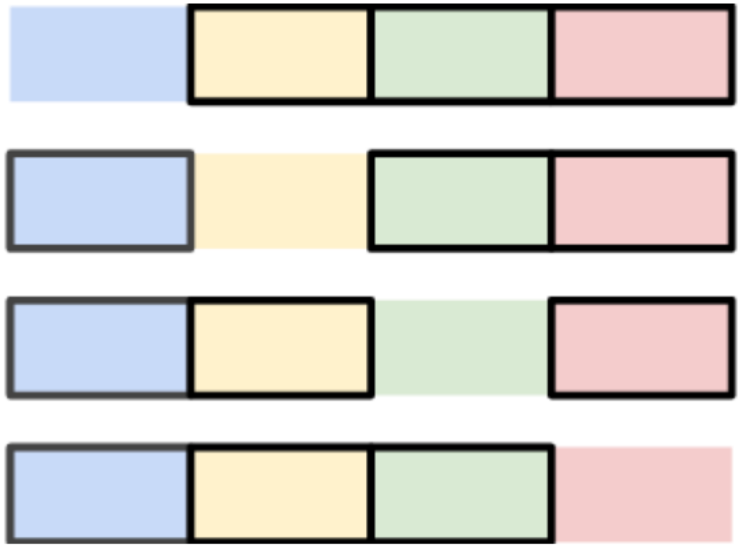
Symmetry of combinations: $\binom{n}{k} = \binom{n}{n-k}$

Two **equivalent** ways to choose **k out of n objects** (unordered):

*** Choose** k elements to include



* Choose n-k elements to exclude



A more formal second proof

Suppose you have n people, and need to choose k people to be on your team. We will count the number of possible teams two different ways.

Way 1: We choose the k people to be on the team. Since order doesn't matter (you're on the team or not), there are $\binom{n}{k}$ possible teams.

Way 2: We choose the $n-k$ people to NOT be on the team. Everyone else is on it. Since order again doesn't matter, there are $\binom{n}{n-k}$ possible ways to choose the team.

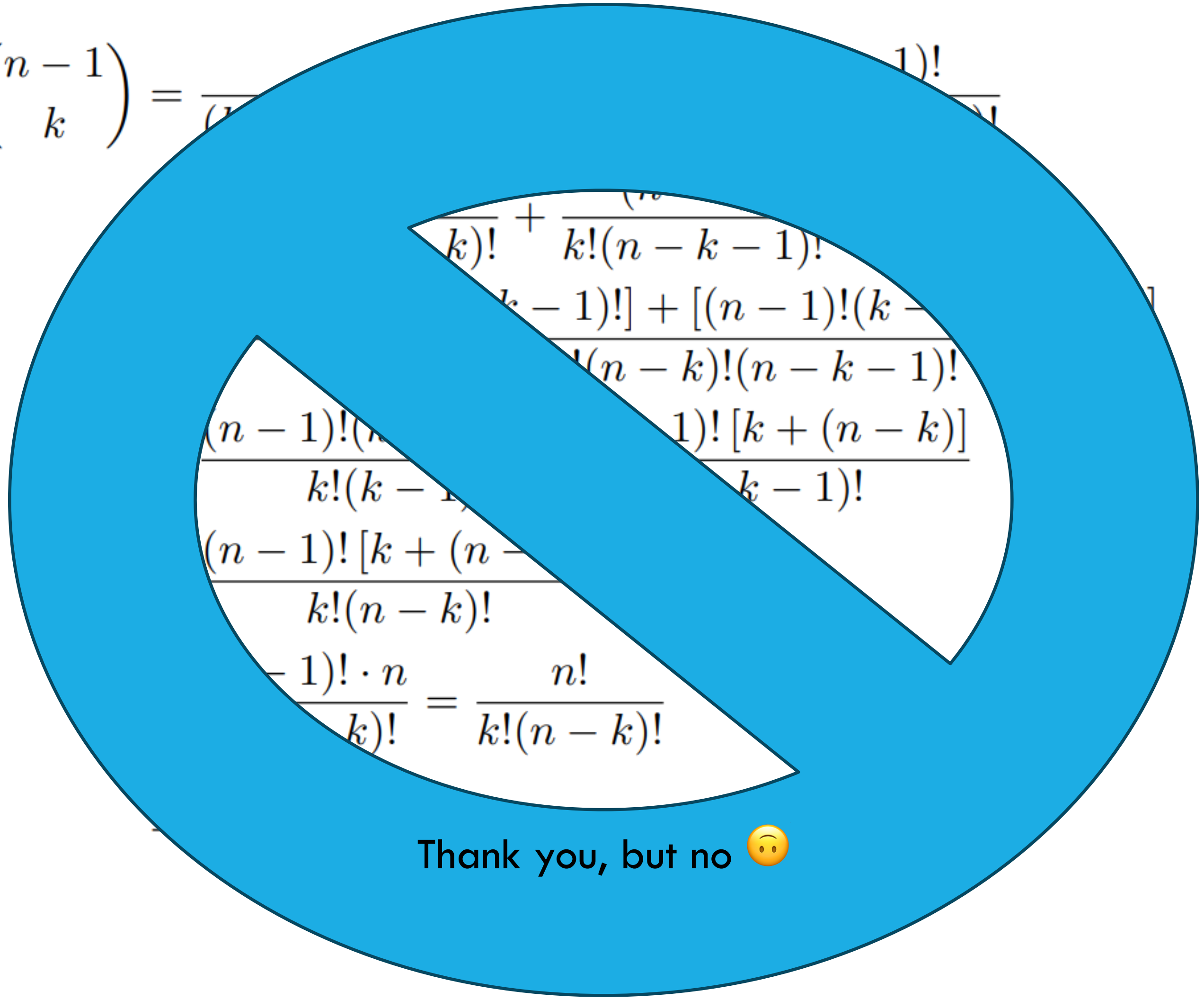
Since we're counting the same thing, the numbers must be equal.
So $\binom{n}{k} = \binom{n}{n-k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$
\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-1-[k-1])!} + \frac{(n-1)!}{k!(n-1-k)!} && \text{definition of combination} \\
&= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} && \text{subtraction} \\
&= \frac{[(n-1)!k!(n-k-1)!] + [(n-1)!(k-1)!(n-k)!]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{Find a common denominator} \\
&= \frac{(n-1)!(k-1)!(n-k-1)!\,[k+(n-k)]}{k!(k-1)!(n-k)!(n-k-1)!} && \text{factor out common terms} \\
&= \frac{(n-1)!\,[k+(n-k)]}{k!(n-k)!} && \text{Cancel } (k-1)!(n-k-1)! \\
&= \frac{(n-1)! \cdot n}{k!(n-k)!} = \frac{n!}{k!(n-k)!} && \text{Algebra} \\
&= \binom{n}{k} && \text{Definition of combination}
\end{aligned}
$$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \cdots$$



$$\cdots = \frac{n!}{k!(n-k)!}$$

definition of combination

subtraction

Find a common denominator

factor out common terms

Cancel $(k-1)!(n-k-1)!$

Algebra

Definition of combination

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

You and $n - 1$ other people are trying out for a k person team. How many possible teams are there?

Way 1: There are n people total, of which we're choosing k (and since it's a team order doesn't matter) $\binom{n}{k}$.

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

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Way 2: There are two ways to make teams: EITHER you are on the team of k people OR you are not on the team of k.

These are disjoint sets – sound familiar?

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

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These are disjoint sets – sound familiar?

- If I'm on the team, the remaining k-1 are chosen from the n-1 people who are not me.
- If I am on the team, all k are chosen from the n-1 people who are not me.

By the sum rule: $\binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Rule: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

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Way 2: : There are two ways to make teams: EITHER you are on the team of k people OR you are not on the team of k.

By the sum rule: $\binom{n-1}{k-1} + \binom{n-1}{k}$

Since we're computing the same number two different ways, they must be equal. So: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Combinatorial Proofs

Let S be a set of objects.

Show how to count |S| one way => |S| = N

Show how to count |S| another way => |S| = M

We are counting the same thing, so both methods of counting are equal (M=N).

Can be much simpler and more intuitive than algebraic proofs :)

Key Takeaways

We know how to count when order matters, and when it doesn't. And we know how to fake it through a trick…

A useful trick for counting is to pretend order matters, then account for the overcounting at the end (by dividing out repetitions)

When trying to prove facts about counting, try to have each side of the equation count the same thing.

Much more fun and much more informative than just churning through algebra.

Extra Practice

Books, revisited

Remember the books problem from lecture 1? Books 1,2,3,4,5 need to be assigned to Alice, Bob, and Charlie (each book to exactly one person).

Now that we know combinations, try a sequential process approach. It won't be as nice as the change of perspective, but we can make it work.

Break into cases based on how many books Alice gets, use the sum rule to combine.

Books, revisited

Step 1: give Alice gets 0 books (1 way to do this)

Step 2: give Bob a subset of the remaining books 2^5 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+$

Step 1: give Alice 1 book ($\binom{5}{1}$ ways to do this)

Step 2: give Bob a subset of the 4 remaining books 2^4 ways.

Step 3: give Charlie the remaining books (no choice – 1 way)

$+\ \dots$

Books, revisited

Add all the options together

$$1 \cdot 2^5 \cdot 1 + \binom{5}{1} \cdot 2^4 \cdot 1 + \binom{5}{2} \cdot 2^3 \cdot 1 + \binom{5}{3} \cdot 2^2 \cdot 1 + \binom{5}{4} \cdot 2^1 \cdot 1 + \binom{5}{5} \cdot 2^0 \cdot 1$$

If you plug and chug, you'll get the number we got last time. It took quite a bit of work, but we got there!

Combinatorial Proof

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

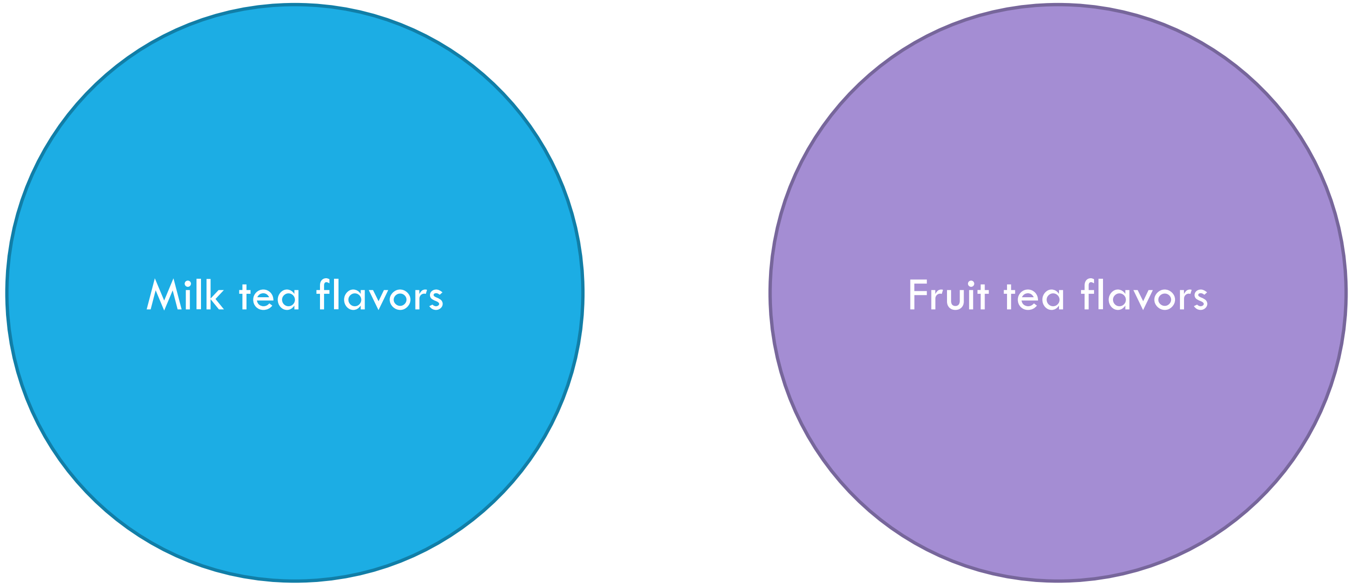
LHS: number of ways to create a subset out of n elements, partitioning on the size of the subset

RHS: for each element, it is either in the subset or not in the subset

Principle of Inclusion-Exclusion

Recall: Sum Rule

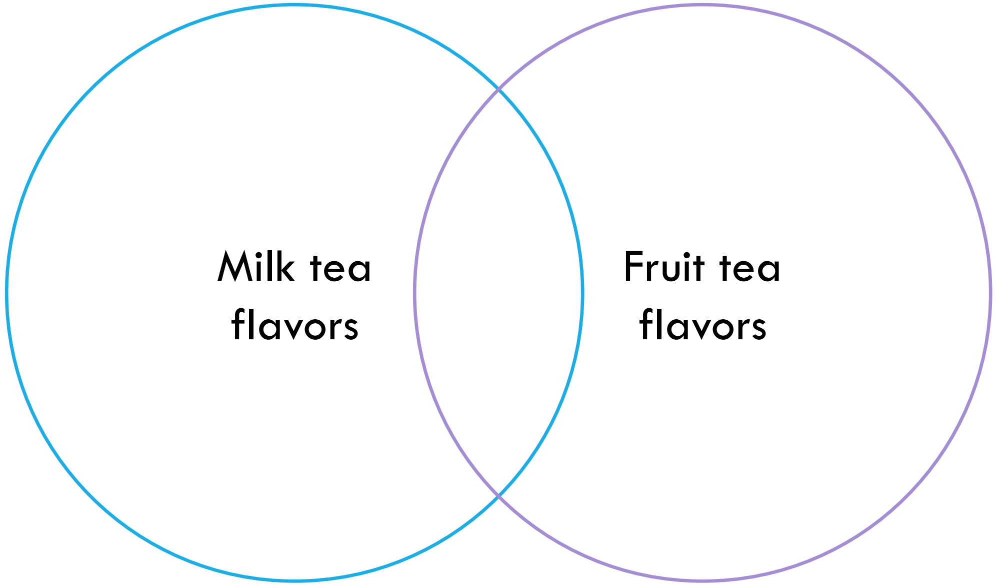
If a boba shop sells 5 milk tea flavors and 8 fruit tea flavors, how many possible orders of 1 drink? => <u>13 possible orders</u>



Sum Rule: If you are choosing one thing between n options in one group and m in another group with no overlap, the total number of options is: $n + m$.

What happens when the groups overlap?

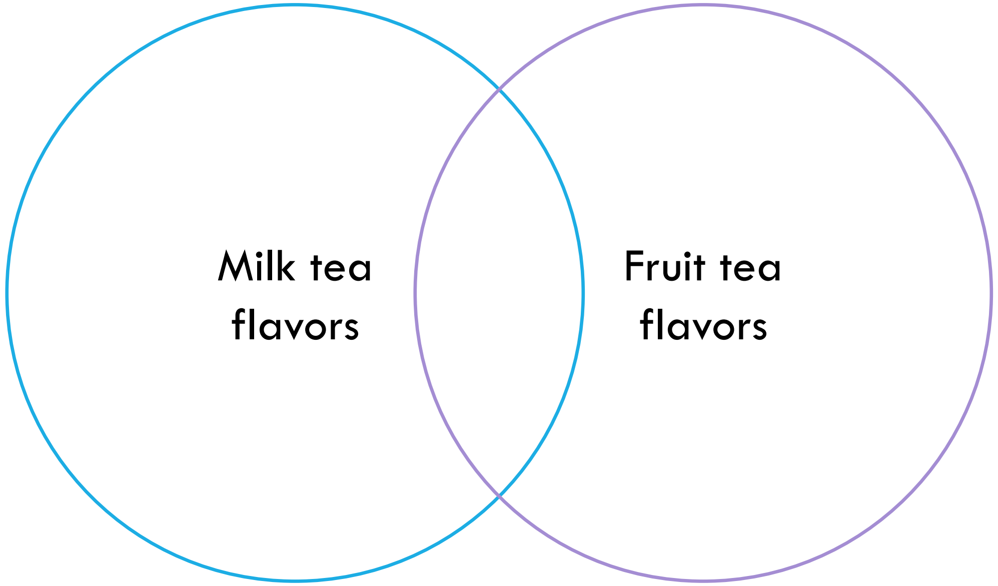
There are 5 milk tea flavors, 8 fruit tea flavors and 3 fall in both categories!



What happens when the groups overlap?

There are 5 milk tea flavors, 8 fruit tea flavors and 3 fall in both categories!

Let's start by adding up the number of milk tea flavors and fruit tea flavors, to get 5+8=13.



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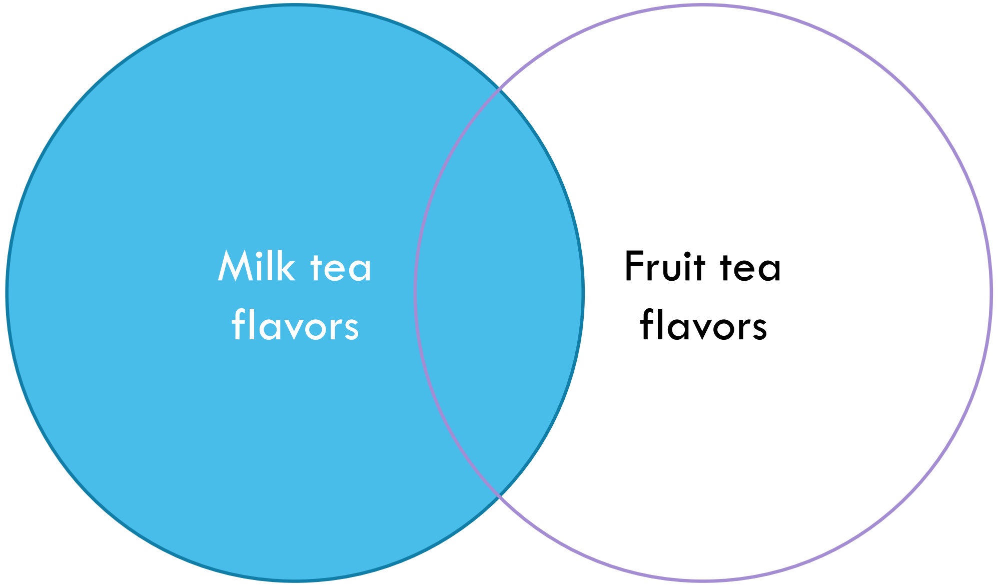
Let's start by adding up the number of milk tea flavors: 5…

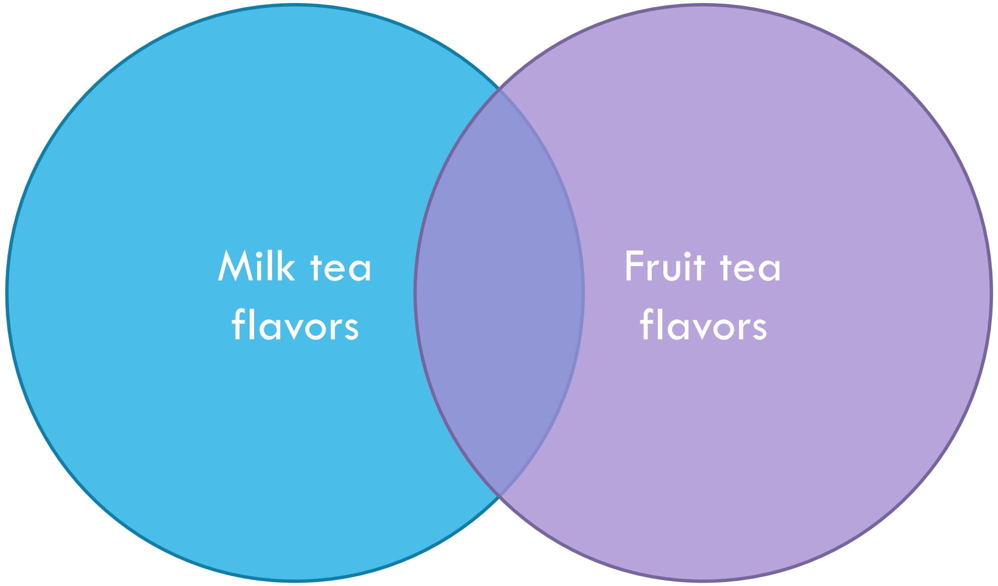


What happens when the groups overlap?

There are 5 milk tea flavors, 8 fruit tea flavors and 3 fall in both categories!

Let's start by adding up the number of milk tea flavors: 5…

And the number of fruit tea flavors: 8…



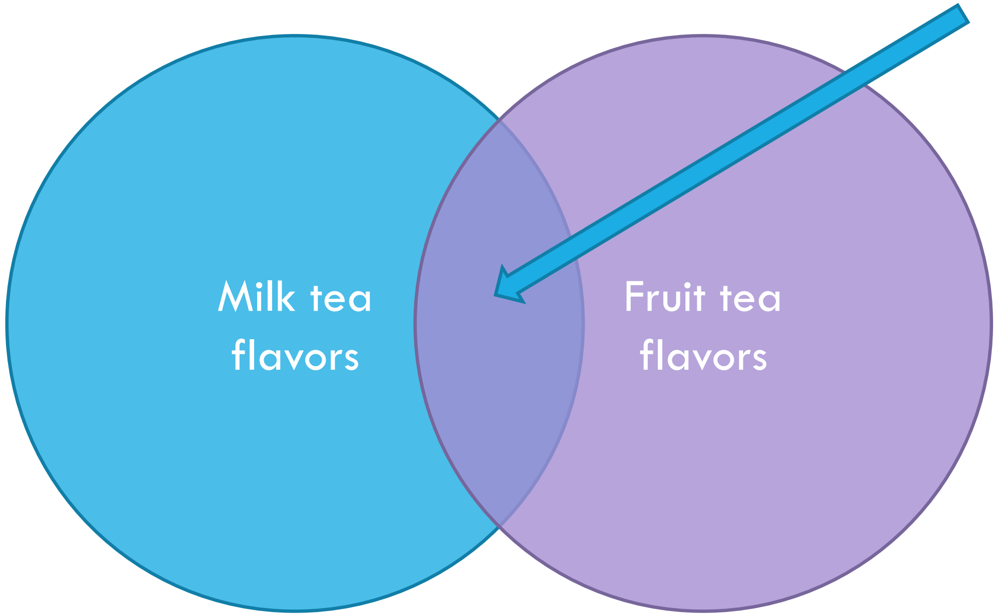
What happens when the groups overlap?

There are 5 milk tea flavors, 8 fruit tea flavors and 3 fall in both categories!

Let's start by adding up the number of milk tea flavors: 5…

And the number of fruit tea flavors: 8…

This region, the intersection of the two sets ($M \cap F$), has been counted twice!



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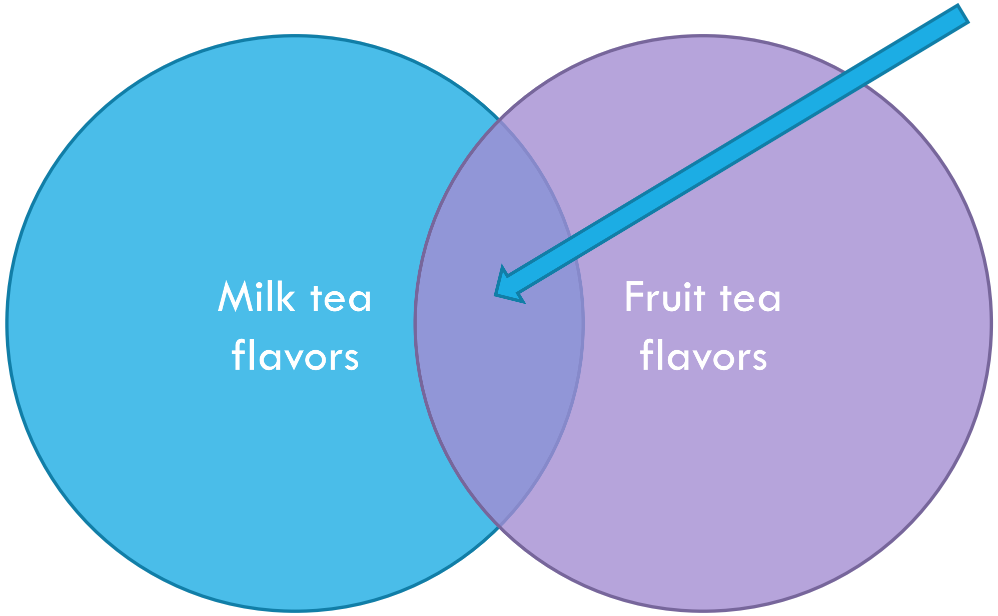


This region, the intersection of the two sets ($M \cap F$), has been counted twice!

We want to count it: 1x. So we subtract $M \cap F$ one time to ensure every tea is counted once.

What happens when the groups overlap?

There are 5 milk tea flavors, 8 fruit tea flavors and 3 fall in both categories!

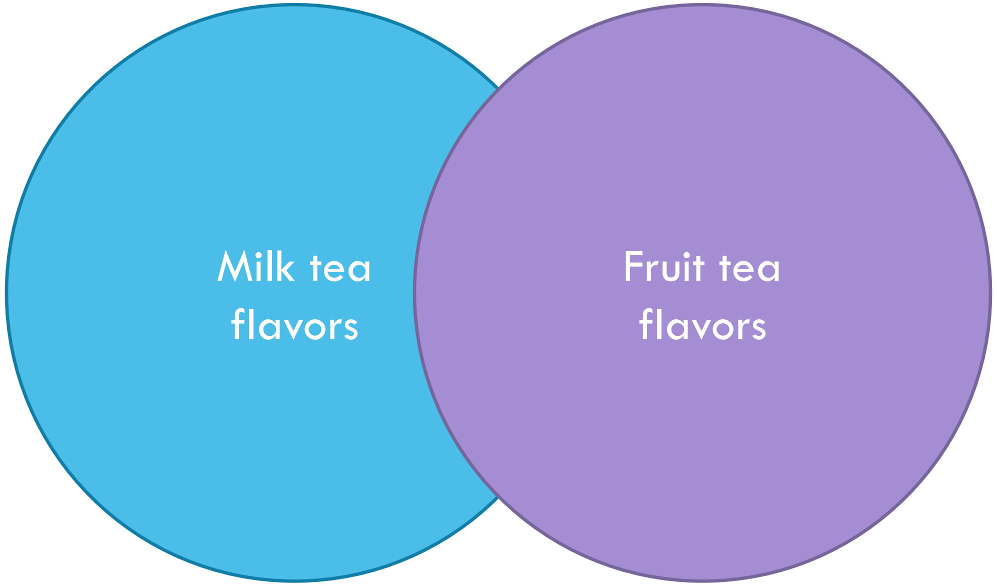
Let's start by adding up the number of milk tea flavors: 5…

And the number of fruit tea flavors: 8…

Now, we subtract the teas that fall in both categories: 3…

Which gives us $|M| + |F| - |M \cap F|$

$= 8 + 5 - 3 = 10$ possible orders now!

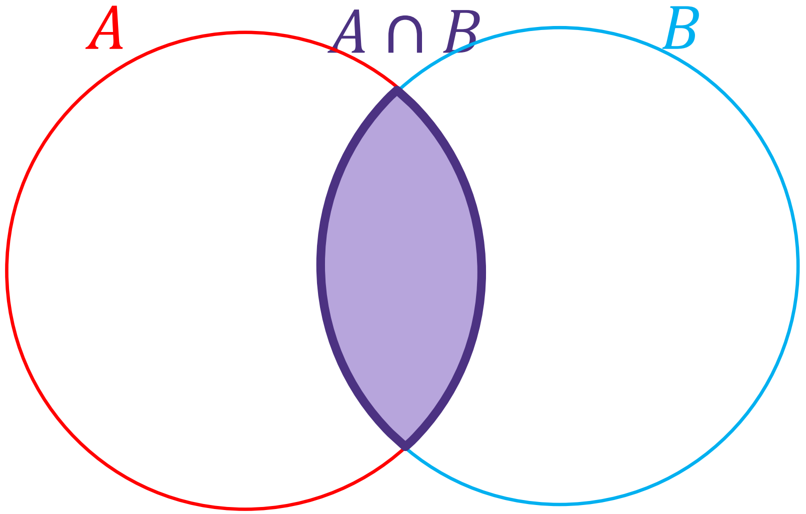


Principle of Inclusion-Exclusion

The sum rule says when A and B are disjoint (no intersection), then $|A \cup B| = |A| + |B|$.

When A and B aren't disjoint, for two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$



An example with 3 sets

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:

Exactly 2 ‘a’s OR

Exactly 1 ‘b’ OR

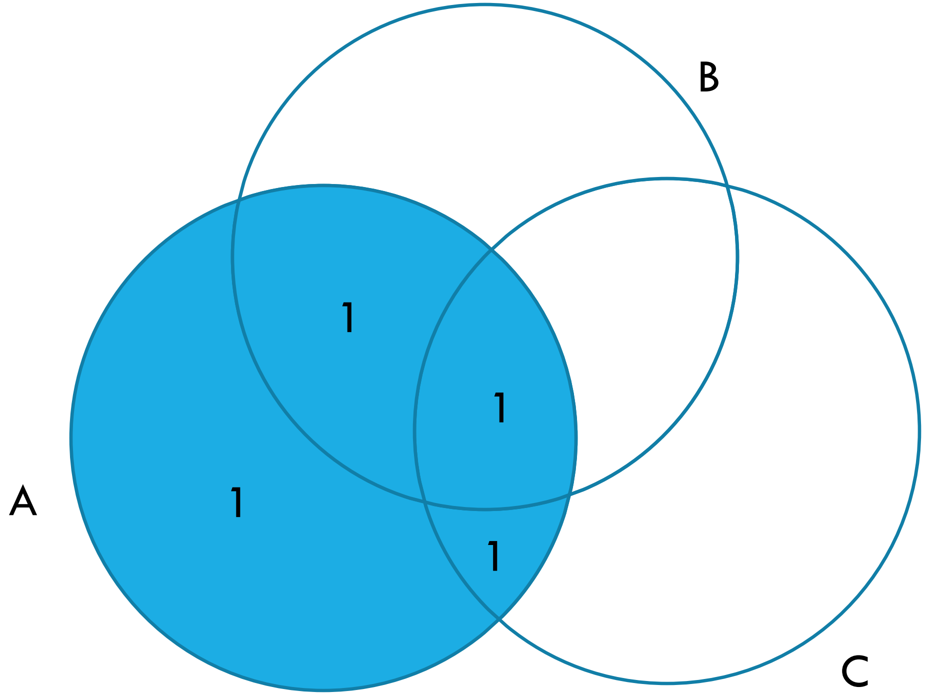
No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$



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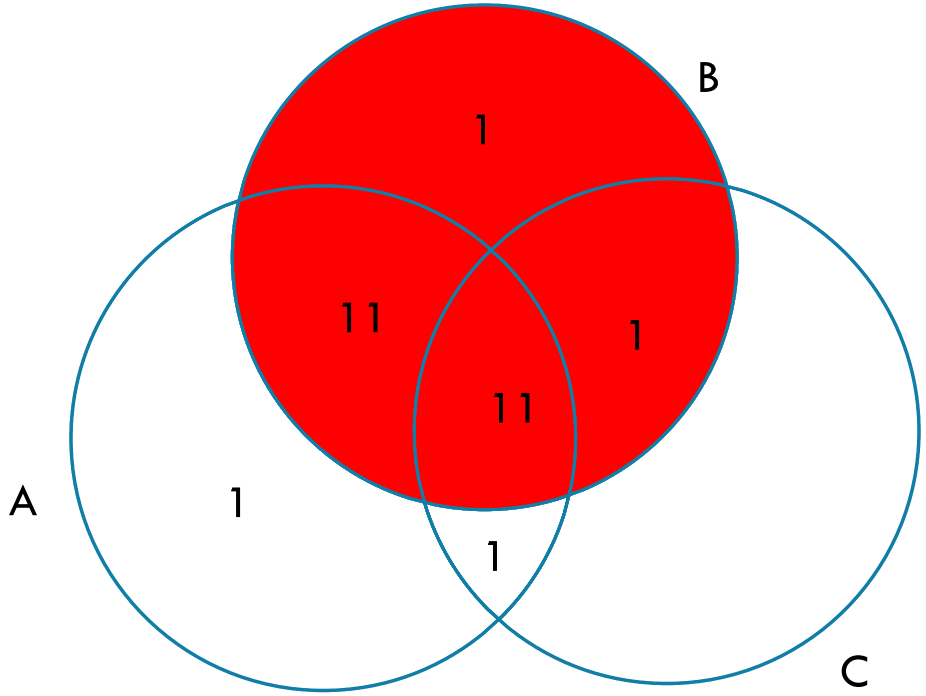

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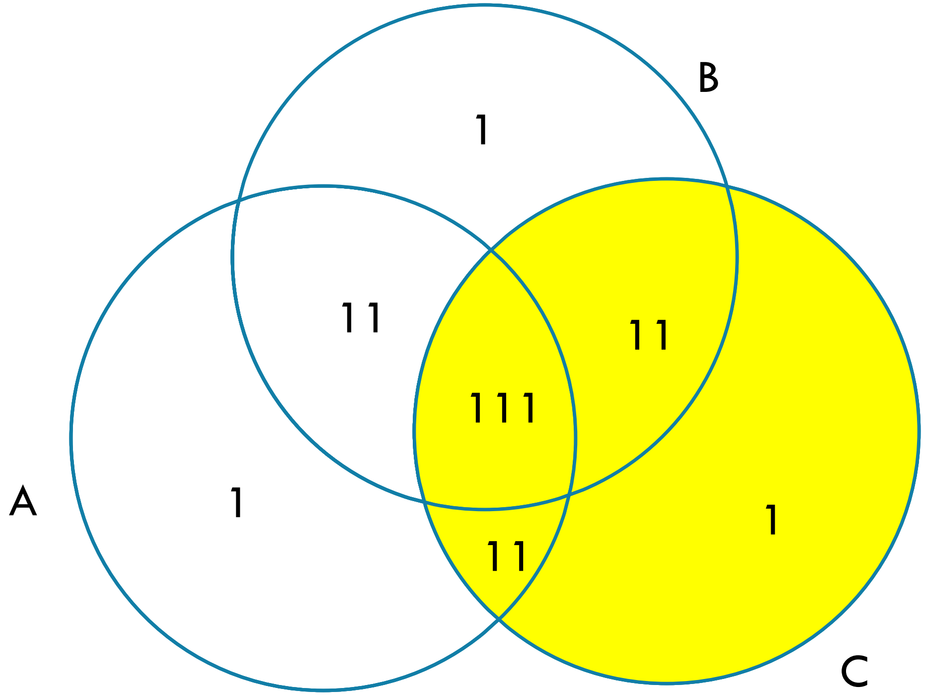

Principle of Inclusion-Exclusion

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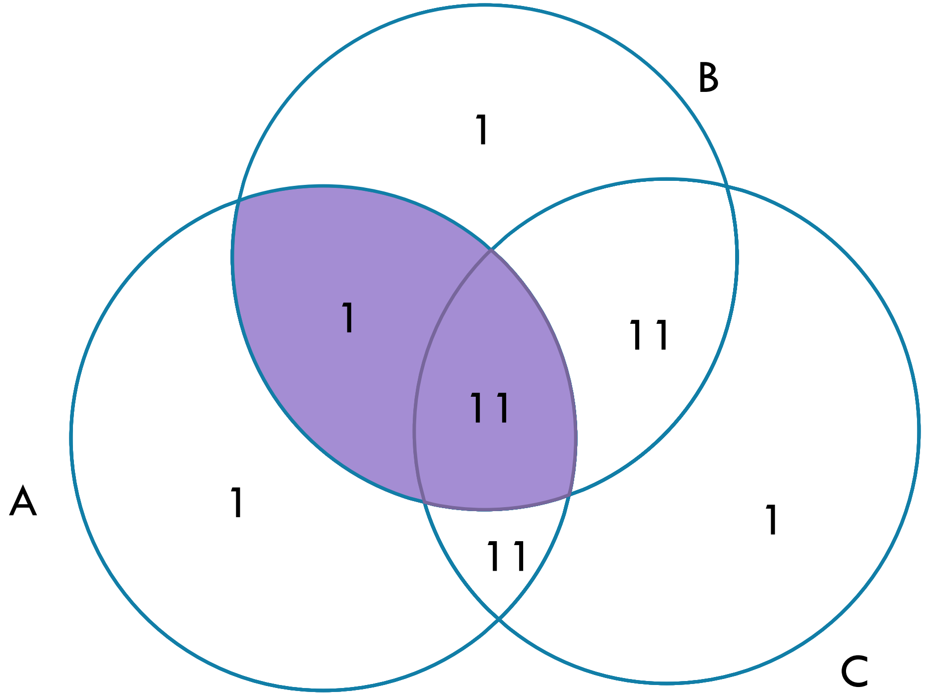

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For three sets:

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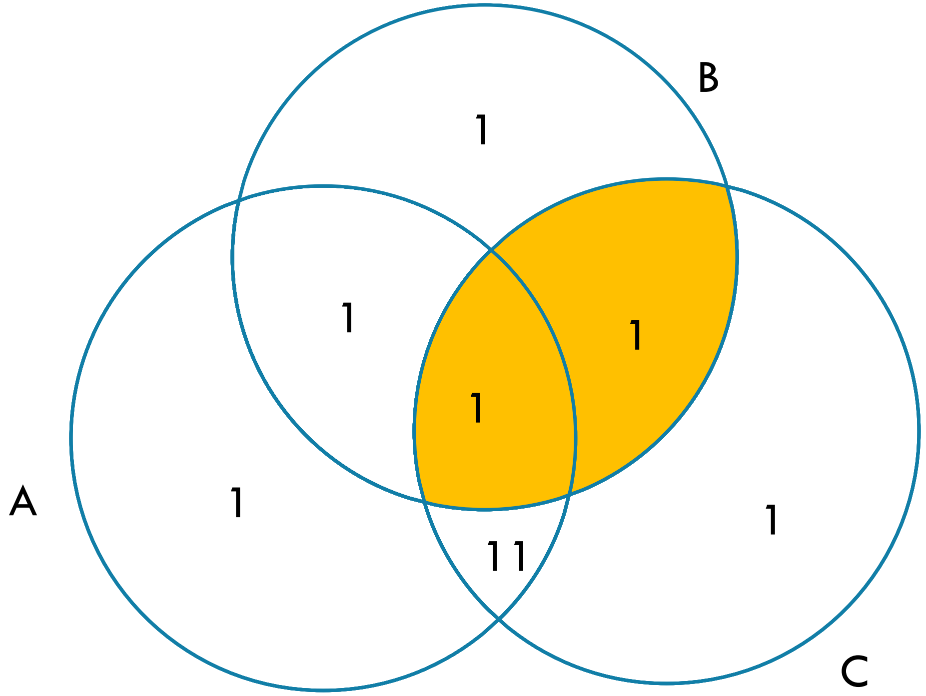

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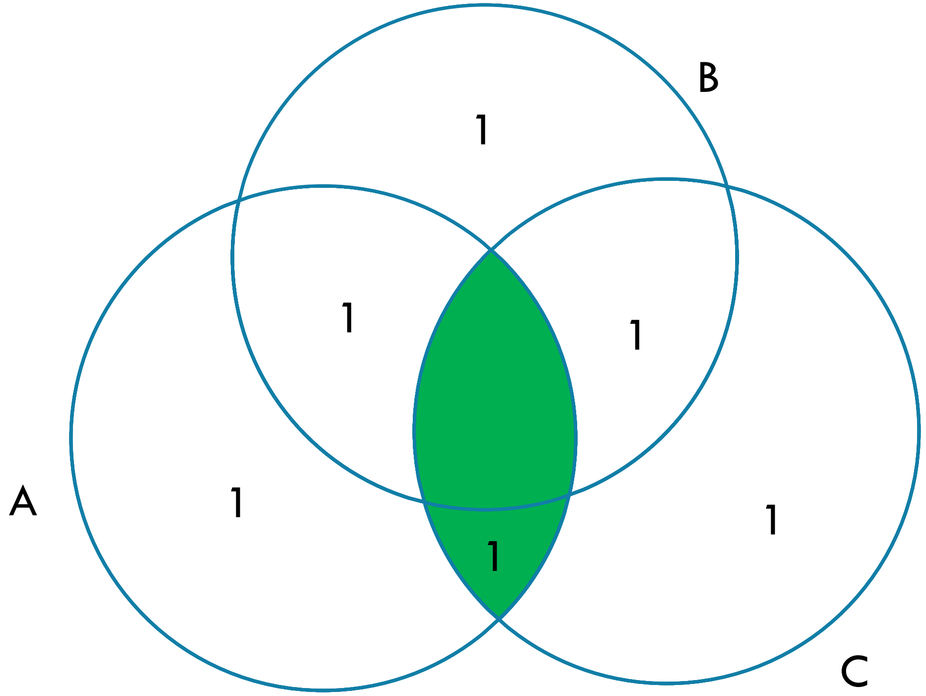
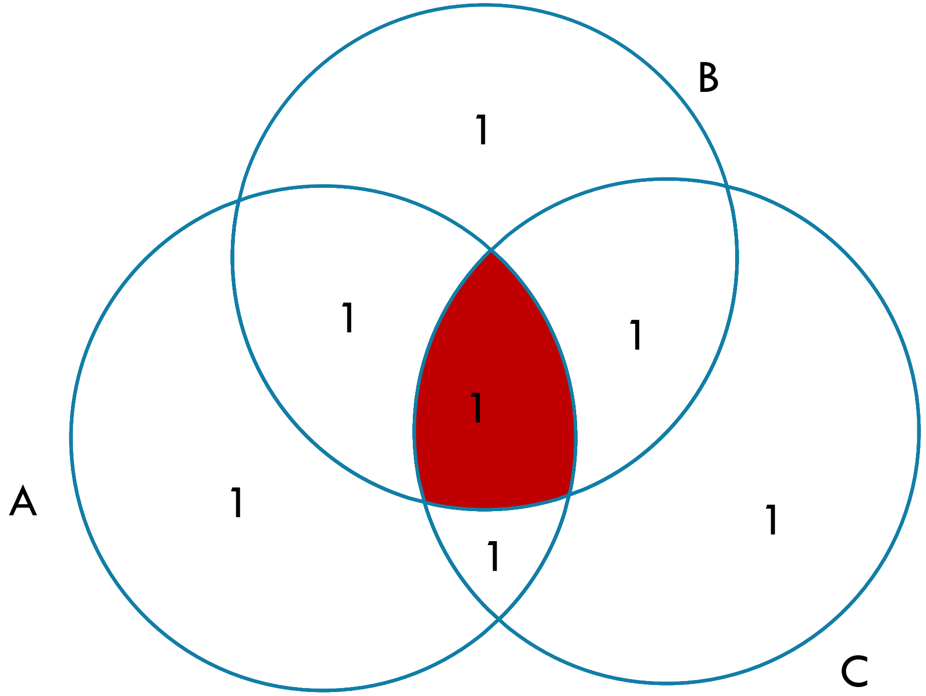


Principle of Inclusion-Exclusion

For three sets:

$$|A \cup B \cup C| = \boxed{|A|} + \boxed{|B|} + \boxed{|C|} - \boxed{|A \cap B|} - \boxed{|B \cap C|} - \boxed{|A \cap C|} + \boxed{|A \cap B \cap C|}$$



In general:

$$
\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_n| = \\
&\quad |A_1| + |A_2| + \cdots + |A_n| \\
&\quad -(|A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_1 \cap A_n| + |A_2 \cap A_3| + \cdots + |A_{n-1} \cap A_n|) \\
&\quad +(|A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) \\
&\quad - \ \ldots \\
&+(-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}
$$

Add the individual sets, subtract all pairwise intersections, add all three-wise intersections, subtract all four-wise intersections,…, [add/subtract] the n-wise intersection.

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:

Exactly 2 ‘a’s OR

Exactly 1 ‘b’ OR

No ‘x’s

For what A, B, C do we want $|A \cup B \cup C|$?

Example

How many length 5 strings over the alphabet $\{a, b, c, \ldots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$A =$ {length 5 strings that contain exactly 2 'a's}

$B =$ {length 5 strings that contain exactly 1 'b's}

$C =$ {length 5 strings that contain no 'x's }

$|A| =$

$|B| =$

$|C| =$

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 ‘a’s OR
Exactly 1 ‘b’ OR
No ‘x’s

$A =$ {length 5 strings that contain exactly 2 ‘a’s}

$B =$ {length 5 strings that contain exactly 1 ‘b’s}

$C =$ {length 5 strings that contain no ‘x’s }

$|A| = \binom{5}{2} \cdot 25^3$ (need to choose which “spots” are ‘a’ and remaining string)

$|B| = \binom{5}{1} \cdot 25^4$

$|C| = 25^5$

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$A = \{\text{length 5 strings that contain exactly 2 'a's}\}$

$B = \{\text{length 5 strings that contain exactly 1 'b's}\}$

$C = \{\text{length 5 strings that contain no 'x's }\}$

$|A| = \binom{5}{2} \cdot 25^3$
$|B| = \binom{5}{1} \cdot 25^3$
$|C| = 25^5$

$|A \cap B| =$
$|A \cap C| =$
$|B \cap C| =$
$|A \cap B \cap C|$

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$A =$ {length 5 strings that contain exactly 2 'a's}
$B =$ {length 5 strings that contain exactly 1 'b's}
$C =$ {length 5 strings that contain no 'x's }

$|A| = \binom{5}{2} \cdot 25^3$
$|B| = \binom{5}{1} \cdot 25^3$
$|C| = 25^5$

$|A \cap B| = \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2$ (choose 'a' spots, 'b' spot, remaining chars)
$|A \cap C| = \binom{5}{2} \cdot 24^3$ (choose 'a' spots, remaining [non-'x'] chars)
$|B \cap C| = \binom{5}{1} \cdot 24^4$
$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$ (choose 'a' spots, 'b' spot, remaining [non-'x'] chars)

Example

How many length 5 strings over the alphabet $\{a, b, c, \dots, z\}$ contain:
Exactly 2 'a's OR
Exactly 1 'b' OR
No 'x's

$|A| = \binom{5}{2} \cdot 25^3$

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$|A \cap B \cap C| = \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2$

$$
\begin{aligned}
|A \cup B \cup C| = \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= \binom{5}{2} \cdot 25^3 + \binom{5}{1} \cdot 25^4 + 25^5 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\
&= 11{,}875{,}000 - \binom{5}{2} \cdot \binom{3}{1} \cdot 24^2 - \binom{5}{2} \cdot 24^3 - \binom{5}{1} \cdot 24^4 + |A \cap B \cap C| \\
&= 11{,}875{,}000 - 1{,}814{,}400 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + |A \cap B \cap C| \\
&= 10{,}060{,}600 + \binom{5}{2} \cdot \binom{3}{1} \cdot 23^2 \\
&= 10{,}060{,}600 + 15{,}870 \\
&= 10{,}076{,}470
\end{aligned}
$$

Practical tips

Give yourself clear definitions of A, B, C.

Make a table of all the formulas you need before you start actually calculating.

Calculate "size-by-size" and incorporate into the total.

Basic check: If (in an intermediate step) you ever:

1. Get a negative value
2. Get a value greater than the prior max by adding (after all the single sets)
3. Get a value less than the prior min by subtracting (after all the pairwise intersections)

Then something has gone wrong.