Homework 5: Continuous Random Variables

In general, your goal in an explanation is to write enough that a student from class who has attended lecture, but not thought about the problem yet, could understand your approach, verify your reasoning, and believe your answer is correct. While we do not usually need to see arithmetic, you must include enough work that in principle one could rederive your answer with only a scientific calculator.

Unless a problem states otherwise, you should leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Remember that you must "select pages" to identify which problem is on which page as part of your gradescope upload.

Submission: You must upload a **pdf** of your written solutions to Gradescope under "HW 5 [Written]". The use of *LaTeX* is *required*.

Due Date: This assignment is due at **11 PM Tuesday August 1**.
You will submit the written problems as a PDF to Gradescope. Please put each numbered problem on its own page of the PDF (this will make selecting pages easier when you submit), and ensure that your PDFs are oriented correctly (e.g. not upside-down or sideways). The coding problem will also be submitted to Gradescope.

Note that there will be **no leeway** with the submission time. Any submission to Gradescope after 11:00:00 PM on the deadline day will be considered late. We recommend submitting a few minutes before the deadline in order to avoid technical issues with submissions.

Collaboration: Please read the full collaboration policy. If you work with others, you must still write up your solution independently and name all of your collaborators somewhere on your assignment.

Full submission instructions are in the course syllabus.

0. Extra Instructions :o

For calculations that require evaluating integrals (unless we indicate otherwise), you must

(a) Show the integral to evaluate (e.g., $\int_0^2 z \cdot 2dz$)

(b) Show an antiderivative and the values to evaluate at (e.g., $z^2|_0^2$)

(c) Plug in the values and simplify (e.g., $2^2 - 0^2 = 4$)

This is not a problem, so nothing needs to be submitted here.

1. PDF or DOCX? [18 points]

Consider the following PDF:

$$f_X(x) = \begin{cases} r(1-4x^3) & \text{if } -\frac{1}{3} \leq x \leq \frac{1}{3} \\ 0 & \text{otherwise} \end{cases}$$

Where r is a constant.

(a) What value of r makes the PDF valid? [5 points]

(b) What is $F_X(k)$? [5 points]

(c) What is $\mathbb{E}[X]$? [4 points]

(d) What is $Var(X)$? [4 points]

2. Fleas get Degrees [18 points]

A flea of negligible size is trapped in a large, spherical, inflated beach ball with radius c. At this moment, it is equally likely to be at any point within the ball. Let X be the distance of the flea from the center of the ball. For X, find

(a) the cumulative distribution function F. [5 points]

(b) the probability density function f. [5 points]

(c) the expected value. [4 points]

(d) the variance. [4 points]

Reminder: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

3. A Penny for Your Darts [12 points]

You throw a dart at a circular target of radius r. Let X be the distance of your dart's hit from the center of the target. You have improved and your aim is such that $X \sim \text{Exponential}(7/r)$. (Note that it is possible for the dart to completely miss the target.)

(a) As a function of r, determine the value m such that $\mathbb{P}(X < m) = \mathbb{P}(X > m)$. [6 points]

(b) For $r = 15$, give the value of m to 3 decimal places. [2 points]

(c) What is the probability that you miss the target completely? Give your answer to 4 decimal places. [4 points]

4. I'm So $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ [12 points]

For each question below, if you need to use the CDF of a normal, be sure to round the z-score to the hundredths-place and use the table linked on the webpage.

You fold a paper airplane whose flight length (in feet) is normally distributed with mean 6 and variance 9.

(a) What is the probability that your airplane flies over 10 feet? [4 points]

(b) What is the probability that your airplane flies between 3 and 14 feet? [4 points]

(c) Your friend folds a paper airplane whose flight length is also normally distributed, but with mean 8 and variance 16. What is the probability that in an airplane competition, your airplane flies further than your friend's? [4 points]

5. Wonderful Form of the Cosmic Order [24 points]

In all of the following use the Central Limit Theorem. You should determine if the continuity correction is necessary, and use it when it is.

(a) You write a new sorting algorithm. The runtime for this algorithm averages $\frac{35}{3}$ seconds, with a standard deviation of 2 seconds. What is the probability that it takes less than 100 seconds to run the algorithm 9 times, with each run independent from the other? Give your answer to 4 decimal places. [8 points]

(b) You open a new bookstore. The average price of each book is $\$17$, and the prices are continuously distributed with a variance of $\$15$. You sell 40 books in a day.

Your rich BFF offers to lend you money for your venture, on the condition that you answer the following question: at least how much money will you make at least 95% of the time? Give your answer to 2 decimal places. [8 points]

(c) You play roulette, with probability $\frac{17}{35}$ of winning on a spin. One fine day, you decide to spend your afternoon gambling, and spin the wheel n times. At least how big should n be to guarantee you will win at least 10 times with probability 90%? [8 points]

6. Practice with uniforms [12 points]

Suppose that $X \sim Unif(0,1)$. Let a, b be constants such that $0 < a < b < 1$.

(a) (3 points) What is $\mathbb{P}(X > b | X > a)$?

(b) (3 point) For any real number y, what is $\mathbb{P}(X \leq y | X > a)$? Be sure you include all cases.

(c) (6 points) Let Y be a random variable such that

$$F_Y(y) = \mathbb{P}(X \leq y | X > a).$$

What is $f_Y(y)$ and what kind of random variable is Y? (Note that when you specify a distribution from our zoo, you must specify both the name of the distribution and the values of the parameters of that distribution.)

7. Rolling in the Dice [12 points]

Suppose that a fair 6-sided die is rolled repeatedly, with each roll independent of the others. Let Z be the number of rolls until (and including) the first time either a 2 or a 3 is rolled, and let W be the number of 6's rolled until the first 2 or 3 is rolled. So, for example if the sequence of die values until the first 2 or 3 is 1,5,4,4,5,6,1,6,2, then Z is 9 and W is 2.

Define

$$p(j) := \begin{cases} \Pr(W = j \mid Z = i) & j \in \{0, 1, \ldots, i-1\} \\ 0 & \text{otherwise} \end{cases}$$

Show that $p(j)$ is the probability mass function of a binomially distributed random variable and determine its parameters n and p.