

Section 7: Continuous Random Variables and the Central Limit Theorem

Review of Main Concepts

- **Closure of the Normal Distribution:** Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Then, $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$. That is, linear transformations of normal random variables are still normal.
- **“Reproductive” Property of Normals:** Let X_1, \dots, X_n be independent normal random variables with $\mathbb{E}[\sum X_i] = \mu_i$ and $Var(X_i) = \sigma_i^2$. Let $a_1, \dots, a_n \in \mathbb{R}$ and $b \in \mathbb{R}$. Then,

$$X = \sum_{i=1}^n (a_i X_i + b) \sim \mathcal{N}\left(\sum_{i=1}^n (a_i \mu_i + b), \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

There’s nothing special about the parameters – the important result here is that the resulting random variable is still normally distributed.

- **Law of Total Probability (Continuous):** A is an event, and X is a continuous random variable with density function $f_X(x)$.

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X=x) f_X(x) dx$$

- **Central Limit Theorem (CLT):** Let X_1, \dots, X_n be iid random variables with $\mathbb{E}[\sum X_i] = \mu$ and $Var(X_i) = \sigma^2$. Let $X = \sum_{i=1}^n X_i$, which has $\mathbb{E}[\sum X] = n\mu$ and $Var(X) = n\sigma^2$. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$, which has $\mathbb{E}[\bar{X}] = \mu$ and $Var(\bar{X}) = \frac{\sigma^2}{n}$. \bar{X} is called the *sample mean*. Then, as $n \rightarrow \infty$, \bar{X} approaches the normal distribution $\mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$. Standardizing, this is equivalent to $Y = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ approaching $\mathcal{N}(0, 1)$. Similarly, as $n \rightarrow \infty$, X approaches $\mathcal{N}(n\mu, n\sigma^2)$ and $Y' = \frac{X - n\mu}{\sigma\sqrt{n}}$ approaches $\mathcal{N}(0, 1)$.

It is no surprise that \bar{X} has mean μ and variance σ^2/n – this can be done with simple calculations. The importance of the CLT is that, for large n , regardless of what distribution X_i comes from, \bar{X} is *approximately normally distributed with mean μ and variance σ^2/n* . Don’t forget the continuity correction, only when X_1, \dots, X_n are discrete random variables.

- **Multivariate: Discrete to Continuous:**

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x, y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x, y) > 0\}$	$\{(x, y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x, y) > 0\}$
Joint CDF	$F_{X,Y}(x, y) = \sum_{t < x, s < y} p_{X,Y}(t, s)$	$F_{X,Y}(x, y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(t, s) ds dt$

1. Will the battery last?

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with expectation 10,000 miles. If the owner wants to take a 5000 mile road trip, what is the probability that she will be able to complete the trip without replacing the battery, given that the car has already been used for 2000 miles?

2. Normal questions

- Let X be a normal random with parameters $\mu = 10$ and $\sigma^2 = 36$. Compute $\mathbb{P}(4 < X < 16)$.
- Let X be a normal random variable with mean 5. If $\mathbb{P}(X > 9) = 0.2$, approximately what is $Var(X)$?

(c) Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that

$$\mathbb{P}(X > c) = 0.10.$$

3. Bad Computer

Each day, the probability your computer crashes is 10%, independent of every other day. Suppose we want to evaluate the computer's performance over the next 100 days.

- (a) Let X be the number of crash-free days in the next 100 days. What distribution does X have? Identify $\mathbb{E}[X]$ and $Var(X)$ as well. Write an exact (possibly unsimplified) expression for $\mathbb{P}(X \geq 87)$.
- (b) Approximate the probability of at least 87 crash-free days out of the next 100 days using the Central Limit Theorem. Use continuity correction.

Important: continuity correction says that if we are using the normal distribution to approximate

$$\mathbb{P}(a \leq \sum_{i=1}^n X_i \leq b)$$

where $a \leq b$ are integers and the X_i 's are i.i.d. **discrete** random variables, then, as our approximation, we should use

$$\mathbb{P}(a - 0.5 \leq Y \leq b + 0.5)$$

where Y is the appropriate normal distribution that $\sum_{i=1}^n X_i$ converges to by the Central Limit Theorem.¹

For more details see pages 209-210 in the book.

4. Continuous Law of Total Probability?

This has not been covered in class yet, but will be soon.

In this exercise, we will extend the law of total probability to the continuous case.

- (a) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ (notice this set has size $n + 1$). Let H be the event that the coin comes up heads. What is $\mathbb{P}(H)$?
- (b) Now suppose $U \sim \text{Uniform}(0,1)$ has the *continuous* uniform distribution over the interval $[0, 1]$. Extend the law of total probability to work for this continuous case. (Hint: you may have an integral in your answer instead of a sum).
- (c) Let's generalize the previous result we just used. Suppose E is an event, and X is a continuous random variable with density function $f_X(x)$. Write an expression for $\mathbb{P}(E)$, conditioning on X .

5. Transformations

This has not been covered in class yet and probably won't be. But if you're interested, please read Section 4.4.

Suppose $X \sim \text{Uniform}(0, 1)$ has the continuous uniform distribution on $(0, 1)$. Let $Y = -\frac{1}{\lambda} \log X$ for some $\lambda > 0$.

¹

The intuition here is that, to avoid a mismatch between discrete distributions (whose range is a set of integers) and continuous distributions, we get a better approximation by imagining that a discrete random variable, say W , is a continuous distribution with density function

$$f_W(x) := p_W(i) \quad \text{when } i - 0.5 \leq x < i + 0.5 \text{ and } i \text{ integer}$$

- (a) What is Ω_Y ?
- (b) First write down $F_X(x)$ for $x \in (0, 1)$. Then, find $F_Y(y)$ on Ω_Y .
- (c) Now find $f_Y(y)$ on Ω_Y (by differentiating $F_Y(y)$ with respect to y . What distribution does Y have?

6. Convolutions

This has not been covered in class. We're not yet sure if we will have time for it, but if you're interested, please read Section 5.5.

Suppose $Z = X + Y$, where $X \perp Y$. (\perp is the symbol for independence. In other words, X and Y are independent.) Z is called the convolution of two random variables. If X, Y, Z are discrete,

$$p_Z(z) = \mathbb{P}(X + Y = z) = \sum_x \mathbb{P}(X = x \cap Y = z - x) = \sum_x p_X(x) p_Y(z - x)$$

If X, Y, Z are continuous,

$$F_Z(z) = \mathbb{P}(X + Y \leq z) = \int_{-\infty}^{\infty} \mathbb{P}(Y \leq z - X | X = x) f_X(x) dx = \int_{-\infty}^{\infty} F_Y(z - x) f_X(x) dx$$

Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

- (a) Find an expression for $\mathbb{P}(X_1 < 2X_2)$ using a similar idea to convolution, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).
- (b) Find s , where $\Phi(s) = \mathbb{P}(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

7. Bitcoin users

There is a population of n people. The number of Bitcoin users among these n people is i with probability p_i , where, of course, $\sum_{0 \leq i \leq n} p_i = 1$. We take a random sample of k people from the population (without replacement). Use Bayes Theorem to derive an expression for the probability that there are i Bitcoin users in the population conditioned on the fact that there are j Bitcoin users in the sample. Let B_i be the event that there are i Bitcoin users in the population and let S_j be the event that there are j Bitcoin users in the sample. Your answer should be written in terms of the p_i 's, i, j, n and k .

8. Another continuous r.v.

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & \text{when } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $E(X) = \frac{3}{5}$, find a and b .

9. Point on a line

A point is chosen at random on a line segment of length L . Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

10. Min and max of i.i.d. random variables

Let X_1, X_2, \dots, X_n be i.i.d. random variables each with CDF $F_X(x)$ and pdf $f_X(x)$. Let $Y = \min(X_1, \dots, X_n)$ and let $Z = \max(X_1, \dots, X_n)$. Show how to write the CDF and pdf of Y and Z in terms of the functions $F_X(\cdot)$ and $f_X(\cdot)$.

11. Round off error

Let X be the sum of 100 real numbers, and let Y be the same sum, but with each number rounded to the nearest integer before summing. If the roundoff errors are independent and uniformly distributed between -0.5 and 0.5 , what is the approximate probability that $|X - Y| > 3$?

12. Tweets

A prolific twitter user tweets approximately 350 tweets per week. Let's assume for simplicity that the tweets are independent, and each consists of a uniformly random number of characters between 10 and 140. (Note that this is a discrete uniform distribution.) Thus, the central limit theorem (CLT) implies that the number of characters tweeted by this user is approximately normal with an appropriate mean and variance. Assuming this normal approximation is correct, estimate the probability that this user tweets between 26,000 and 27,000 characters in a particular week. (This is a case where continuity correction will make virtually no difference in the answer, but you should still use it to get into the practice!).

13. Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

- Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).
- Find the marginal PMF for X , $p_X(x)$ for $x \in \Omega_X$.
- Find the marginal PMF for Y , $p_Y(y)$ for $y \in \Omega_Y$.
- Are X and Y independent? Why or why not?
- Find $\mathbb{E}[X^3Y]$.