Review Probability

**Definition.** A **sample space** $\Omega$ is the set of all possible outcomes of an experiment.

**Examples:**
- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

**Definition.** An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

**Examples:**
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die: $E = \{2, 4, 6\}$
**Review Probability space**

**Definition.** A (discrete) **probability space** is a pair \((\Omega, P)\) where:

- \(\Omega\) is a set called the **sample space**.
- \(P\) is the **probability measure**, a function \(P: \Omega \to \mathbb{R}\) such that:
  - \(P(x) \geq 0\) for all \(x \in \Omega\)
  - \(\sum_{x \in \Omega} P(x) = 1\)

Some outcome must show up. Normalized to sum up to 1.

The likelihood (or probability) of each outcome is non-negative.

Either finite or infinite countable (e.g., integers)

Set of possible elementary outcomes

\[ A \subseteq \Omega: P(A) = \sum_{x \in A} P(x) \]

Specify Likelihood (or probability) of each elementary outcome
Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples
Conditional Probability (Idea)

What’s the probability that someone likes ice cream given they like donuts?

\[
\frac{7}{7 + 13} = \frac{7}{20}
\]
Conditional Probability

**Definition.** The **conditional probability** of event $A$ **given** an event $B$ happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An equivalent and useful formula is

$$P(A \cap B) = P(A|B)P(B)$$

$$\frac{|A \cap B|}{|\Omega|} = \frac{|A \cap B|}{|B|} \cdot \frac{|B|}{|\Omega|}$$
Conditional Probability Examples

Suppose that you flip a fair coin twice. What is the probability that both flips are heads given that you have at least one head?

Let $O$ be the event that at least one flip is heads
Let $B$ be the event that both flips are heads

$P(O) = 3/4$  $P(B) = 1/4$  $P(B \cap O) = 1/4$

$P(B|O) = \frac{P(B \cap O)}{P(O)} = \frac{1/4}{3/4} = \frac{1}{3}$
Conditional Probability Examples

Suppose that you flip a fair coin twice. What is the probability that at least one flip is heads given that at least one flip is tails?

Let \( H \) be the event that at least one flip is *heads*.
Let \( T \) be the event that at least one flip is *tails*.
Conditional Probability Examples

Suppose that you flip a fair coin twice. What is the probability that at least one flip is heads given that at least one flip is tails?

Let $H$ be the event that at least one flip is heads
Let $T$ be the event that at least one flip is tails

$P(H) = \frac{3}{4}$  $P(T) = \frac{3}{4}$  $P(H \cap T) = \frac{1}{2}$

$$P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{1/2}{3/4} = \frac{2}{3}$$
Example with Conditional Probability

Suppose we toss a red die and a blue die: both 6 sided and all outcomes equally likely. What is \( P(B) \)? What is \( P(B | A) \)?

\[
\begin{array}{c|c|c}
\text{ } & P(B) & P(B | A) \\
\hline
\text{a) } & \frac{1}{6} & \frac{1}{6} \\
\text{b) } & \frac{1}{6} & \frac{1}{3} \\
\text{c) } & \frac{1}{6} & \frac{3}{36} \\
\text{d) } & \frac{1}{9} & \frac{1}{3}
\end{array}
\]

\( P(A) = \frac{3}{36} = \frac{1}{12} \)

\[
P(B | A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{12}}{\frac{3}{36}} = \frac{1}{3}
\]

\( A \cap B = \{(1, 3)\} \)

\[
P(A \cap B) = \frac{1}{36}
\]
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- Bayes Theorem
- Law of Total Probability
- More Examples
Reversing Conditional Probability

**Question:** Does $P(A|B) = P(B|A)$?

No!

- Let $A$ be the event you are wet
- Let $B$ be the event you are swimming

\[ P(A|B) = 1 \quad P(B|A) \neq 1 \]
Bayes Theorem

A formula to let us “reverse” the conditional.

**Theorem. (Bayes Rule)** For events $A$ and $B$, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning $B$)
Bayes Theorem Proof

Claim:

\[ P(A), P(B) > 0 \quad \Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]
Bayes Theorem Proof

By definition of conditional probability
\[ P(A \cap B) = P(A|B)P(B) \]

Swapping \( A, B \) gives
\[ P(B \cap A) = P(B|A)P(A) \]

But \( P(A \cap B) = P(B \cap A) \), so
\[ P(A|B)P(B) = P(B|A)P(A) \]

Dividing both sides by \( P(B) \) gives
\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]
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Partitions (Idea)

These events **partition** the sample space
1. They “cover” the whole space
2. They don’t overlap
**Partition**

**Definition.** Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space $\Omega$ if

(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

(Pairwise Mutually Exclusive)

$$\forall_i \forall_{i \neq j} E_i \cap E_j = \emptyset$$
Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition $\Omega$, what can we say about $P(F)$?
Law of Total Probability (LTP)

**Definition.** If events $E_1, E_2, ..., E_n$ partition the sample space $\Omega$, then for any event $F$

$$P(F) = P(F \cap E_1) + ... + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$

We can get the alternate form of this that shows

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$
Another Contrived Example

Alice has two pockets:
• **Left pocket:** Two blue balls, two green balls
• **Right pocket:** One blue ball, two green balls.

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]
Sequential Process

\[ \Pr(B) = \Pr(B \cap \text{Left}) + \Pr(B \cap \text{Right}) \]
\[ = \Pr(\text{Left}) \times \Pr(B|\text{Left}) + \Pr(\text{Right}) \times \Pr(B|\text{Right}) \]
\[ = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12} \]

- **Left pocket**: Two blue, two green
- **Right pocket**: One blue, two green

\[ \frac{1}{3} = \Pr(B|\text{Right}) \] and \[ \frac{2}{3} = \Pr(G|\text{Right}) \] (Law of total probability)
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Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns “positive”, what is the likelihood that you actually have the disease?

• Tests for diseases are rarely 100% accurate.
Example – Zika Testing

Suppose we know the following Zika stats
- A test is 98% effective at detecting Zika (“true positive”)
- However, the test may yield a “false positive” 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event $Z$) if you test positive (event $T$)?

$P(Z)$

$P(T|Z)$

$P(T|Z^c)$

$P(Z|T)$
Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika (“true positive”)
- However, the test may yield a “false positive” 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event $Z$) if you test positive (event $T$)?

By Bayes Rule,

$$P(Z|T) = \frac{P(T|Z)P(Z)}{P(T)}$$

By the Law of Total Probability,

$$P(T) = P(T|Z)P(Z) + P(T|Z^c)P(Z^c)$$

$$= \frac{98}{100} \cdot \frac{5}{100} + \frac{1}{100} \cdot \frac{995}{100} = \frac{490}{10000} + \frac{995}{10000}$$

What is the probability that you do not have Zika (event $Z^c$)?

$$P(Z^c) = 1 - P(Z) = 99.5\%$$

So, $P(Z|T) \approx 33\%$
Philosophy – Updating Beliefs

Your beliefs changed drastically

\(Z = \) you have Zika
\(T = \) you test positive for Zika

I have a 0.5% chance of having Zika

Prior: \(P(Z)\)

Receive positive test result

I now have a 33% chance of having Zika after the test!!

Posterior: \(P(Z|T)\)
Example – Zika Testing

Suppose we know the following Zika stats
  – A test is 98% effective at detecting Zika (“true positive”) \( P(T|Z) \)
  – However, the test may yield a “false positive” 1% of the time \( P(T|Z^c) \)
  – 0.5% of the US population has Zika. \( P(Z) \)

What is the probability you have Zika (event \( Z \)) if you test negative (event \( T^c \))?  

By Bayes Rule, \( P(Z|T^c) = \frac{P(T^c|Z)P(Z)}{P(T^c)} \)

By the Law of Total Probability, \( P(T^c) = P(T^c|Z)P(Z) + P(T^c|Z^c)P(Z^c) \)
= \( \frac{2}{100} \cdot \frac{5}{1000} + \left(1 - \frac{1}{100}\right) \cdot \frac{995}{1000} = \frac{10}{100000} + \frac{98505}{100000} \)

What is the probability you test negative (event \( T^c \)) if you have Zika (event \( Z \))?  
\( P(T^c|Z) = 1 - P(T|Z) = 2\% \)

So, \( P(Z|T^c) = \frac{10}{10+98505} \approx 0.01\% \)
Bayes Theorem with Law of Total Probability

**Bayes Theorem with LTP:** Let $E_1, E_2, ..., E_n$ be a partition of the sample space, and $F$ and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{\sum_{i=1}^{n} P(F|E_i)P(E_i)}{P(F)}$$

**Simple Partition:** In particular, if $E$ is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$
Bayes Theorem with Law of Total Probability

**Bayes Theorem with LTP:** Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and $F$ and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

**Simple Partition:** In particular, if $E$ is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

We just used this implicitly on the negative Zika test example with $E = Z$ and $F = T^c$
Conditional Probability Defines a Probability Space

The probability conditioned on $\mathcal{A}$ follows the same properties as (unconditional) probability.

**Example.** $P(\mathcal{B}^c | \mathcal{A}) = 1 - P(\mathcal{B} | \mathcal{A})$

**Formally.** $(\Omega, P)$ is a probability space and $P(\mathcal{A}) > 0$

$(\mathcal{A}, P(\cdot | \mathcal{A}))$ is a probability space