CSE 312
Foundations of Computing II

Lecture 3: Even more counting
Binomial Theorem, Inclusion-Exclusion, Pigeonhole Principle
Recap

Two core rules for counting a set $S$:

• **Sum rule:**
  – Break up $S$ into disjoint pieces/cases
  – $|S| = \text{the sum of the sizes of the pieces.}$

• **Product rule:**
  – View the elements of $S$ as being constructed by a series of choices, where the # of possibilities for each choice doesn’t depend on the previous choices
  – $|S| = \text{the product of the # of choices in each step of the series.}$
Recap

- **$k$-sequences**: How many length $k$ sequences over alphabet of size $n$?
  - Product rule $\Rightarrow n^k$

- **$k$-permutations**: How many length $k$ sequences over alphabet of size $n$, **without repetition**?
  - Permutation $\Rightarrow \frac{n!}{(n-k)!}$

- **$k$-combinations**: How many size $k$ subsets of a set of size $n$ (**without repetition and without order**)?
  - Combination $\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$
Binomial Coefficients – Many interesting and useful properties

\[
\binom{n}{k} = \frac{n!}{k! (n - k)!}
\]

\[
\binom{n}{n} = 1 \quad \binom{n}{1} = n \quad \binom{n}{0} = 1
\]

**Fact.** \(\binom{n}{k} = \binom{n}{n-k}\)  
Symmetry in Binomial Coefficients

**Fact.** \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\)  
Follows from Binomial Theorem

**Fact.** \(\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\)  
Pascal’s Identity
Agenda

• Binomial Theorem
• Combinatorial Proofs for Pascal Identity
• Inclusion-Exclusion
• Pigeonhole Principle
• Counting Practice
Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

How to prove Pascal’s identity?

Algebraic argument:

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)! (n-k)!} + \frac{(n-1)!}{k! (n-1-k)!}
= \frac{n!}{k! (n-k)!}
= \binom{n}{k}
\]

Hard work and not intuitive

Let’s see a combinatorial argument
Example – Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[
|S| = |B| + |A|
\]

Combinatorial proof idea:
• Find disjoint sets \( A \) and \( B \) such that \( A, B, \) and \( S = A \cup B \) have the sizes above.
• The equation then follows by the Sum Rule.

One Minute Discussion
Example – Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[ |S| = |B| + |A| \]

\( S \): set of size \( k \) subsets of \( [n] = \{1, 2, \ldots, n\} \).

e.g. \( n = 4, k = 2, \quad S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} \)

\( A \): set of size \( k \) subsets of \( [n-1] \) (i.e., DON’T include \( n \))

\[ A = \{\{1,2\}, \{1,3\}, \{2,3\}\} \]

\( B \): Choose a size \( k - 1 \) subsets of \( [n-1] \) then add \( n \) (i.e., DO include \( n \))

\[ B = \{\{1,4\}, \{2,4\}, \{3,4\}\} \]

Combinatorial proof idea:
- Find disjoint sets \( A \) and \( B \) such that \( A, B, \) and \( S = A \cup B \) have these sizes.
Example – Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[ |S| = |B| + |A| \]

\( S \): set of size \( k \) subsets of \([n] = \{1, 2, \ldots, n\} \).

\( A \): set of size \( k \) subsets of \([n - 1]\) (i.e., DON’T include \( n\)).

\( B \): Choose a size \( k - 1 \) subsets of \([n - 1]\) then add \( n\).

Combinatorial proof idea:
• Find disjoint sets \( A \) and \( B \) such that \( A, B, \) and \( S = A \cup B \) have these sizes

\( n \) not in set, need to choose \( k \) elements from \([n - 1]\)

\[ |B| = \binom{n-1}{k} \]

\( n \) is in set, need to choose other \( k - 1 \) elements from \([n - 1]\)

\[ |A| = \binom{n-1}{k-1} \]
Agenda

- Binomial Theorem
- Combinatorial Proofs
- Inclusion-Exclusion
- Pigeonhole Principle
- Counting Practice
Recap Disjoint Sets

Sets that do not contain common elements \((A \cap B = \emptyset)\)

Sum Rule: \(|A \cup B| = |A| + |B|\)
Inclusion-Exclusion

But what if the sets are not disjoint?

\[ A \cup B = A + B - |A \cap B| \]

\[ |A| = 43 \]
\[ |B| = 20 \]
\[ |A \cap B| = 7 \]
\[ |A \cup B| = ??? \]

Fact. \[ |A \cup B| = |A| + |B| - |A \cap B| \]
What if there are three sets?

\[ |A| = 43 \]
\[ |B| = 20 \]
\[ |C| = 35 \]
\[ |A \cap B| = 7 \]
\[ |A \cap C| = 16 \]
\[ |B \cap C| = 11 \]
\[ |A \cap B \cap C| = 4 \]
\[ |A \cup B \cup C| = \text{???} \]

**Fact.**

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
Inclusion-Exclusion

Let $A, B$ be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, ..., A_n$ are sets, then

$$|A_1 \cup A_2 \cup ... \cup A_n| = \text{singles} - \text{doubles} + \text{triples} - \text{quads} + ...$$

$$= (|A_1| + ... + |A_n|) - (|A_1 \cap A_2| + ... + |A_{n-1} \cap A_n|) + ...$$
Agenda

• Binomial Theorem
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• Pigeonhole Principle
• Counting Practice
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes
Pigeonhole Principle: Idea

If 11 children have to share 3 cakes, at least one cake must be shared by how many children?
Pigeonhole Principle – More generally

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

**Proof.** Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k \cdot \frac{n}{k} = n$ pigeons overall.

Contradiction!
Pigeonhole Principle – Better version

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can’t have fractional number of pigeons

Syntax reminder:
• Ceiling: $[x]$ is $x$ rounded up to the nearest integer (e.g., $[2.731] = 3$)
• Floor: $[x]$ is $x$ rounded down to the nearest integer (e.g., $[2.731] = 2$)
Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP
Pigeonhole Principle – Example

*In a room with 367 people, there are at least two with the same birthday.*

Solution:
1. **367** pigeons = people
2. **366** holes (365 for a normal year + Feb 29) = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday
Pigeonhole Principle – Example (Surprising?)

In every set $S$ of 100 integers, there are at least two elements whose difference is a multiple of 37.

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

One Minute Discussion
Pigeonhole Principle – Example (Surprising?)

**In every set $S$ of 100 integers, there are at least two elements whose difference is a multiple of 37.**

When solving a PHP problem:
1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeons: integers $x$ in $S$
Pigeonholes: $\{0,1,\ldots,36\}$

Assignment: $x$ goes to $x \mod 37$

Since $100 > 37$, by PHP, there are $x \neq y \in S$ s.t.

$x \mod 37 = y \mod 37$ which implies

$x - y = 37k$ for some integer $k$
Agenda

• Binomial Theorem
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• Pigeonhole Principle
• Counting Practice
Quick Review of Cards

- 52 total cards
- 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

How many possible 5 card hands?

\[
\binom{52}{5}
\]
A "straight" is five consecutive rank cards of any suit (where A,2,3,4,5 also counts as consecutive).
How many possible straights?

\[ 10 \cdot 4^5 = 10,240 \]
Counting Cards II

- A flush is five card hand all of the same suit. How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]
Counting Cards III

- A flush is five card hand all of the same suit. How many possible flushes?
  \[ 4 \cdot \binom{13}{5} = 5148 \]

- How many flushes are NOT straights?
  \[ = \#\text{flush} - \#\text{flush and straight} \]
  \[ (4 \cdot \binom{13}{5} = 5148) - 10 \cdot 4 \]
For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence $\Rightarrow$ under counting  
Many sequences $\Rightarrow$ over counting

**EXAMPLE:** How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. 

$$\binom{4}{3} \cdot \binom{49}{2}$$

**Poll:**
A. Correct  
B. Overcount  
C. Undercount

https://pollev.com/paulbeame028
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

\[ \binom{4}{3} \cdot \binom{49}{2} \]

Problem: This counts a hand with all 4 Aces in 4 different ways!

* e.g. it counts A♣, A♦, A♥, A♠, 2♥ four times:

\{A♣, A♦, A♥\} \{A♠, 2♥\}
\{A♣, A♦, A♠\} \{A♥, 2♥\}
\{A♥, A♦, A♠\} \{A♣, 2♥\}
\{A♦, A♥, A♠\} \{A♣, 2♥\}
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence $\Rightarrow$ under counting  Many sequences $\Rightarrow$ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

$= \# \text{ 5 card hand containing exactly 3 Aces} + \# \text{ 5 card hand containing exactly 4 Aces}$

$= \binom{4}{3} \cdot \binom{48}{2} + \binom{48}{1}$
Counting when order only *partly* matters

We often want to count # of partly ordered lists:

Let \( M \) = # of ways to produce fully ordered lists

\( P = \) # of partly ordered lists

\( N = \) # of ways to produce corresponding fully ordered list given a partly ordered list

Then \( M = P \cdot N \) by the product rule. Often \( M \) and \( N \) are easy to compute:

\[ P = \frac{M}{N} \]

Dividing by \( N \) “removes” part of the order.
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

**Fully ordered: Pretend Rooks are different**
1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

“Remove” the order of the two rooks:

\[(8 \cdot 7)^2 / 2\]
Binomial Theorem

**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Corollary.**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Set $x = y = 1$
**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- $= -1$ if $k$ is odd
- $= +1$ if $k$ is even

**Corollary.** For every $n$, if $O$ and $E$ are the sets of odd and even integers between 0 and $n$

$$\sum_{k \in O} \binom{n}{k} = \sum_{k \in E} \binom{n}{k}$$

e.g., $n=4$: 1 4 6 4 1

**Proof:** Set $x = -1$, $y = 1$ in the binomial theorem
Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Binary encoding/stars and bars