CSE 312: Foundations of Computing II

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Lecture Topics: 2.1 Discrete Probability, 2.2 Conditional Probability

[**Tags**: Bayes Theorem, Law of Total Probability]

1. Sometimes, doctors run tests to see if we have a disease, but they might not be perfectly accurate. Suppose we are testing for the llama-flu, a highly contagious new disease. So far, only 0.1% of the population has it. If you have llama-flu, the probability the test is negative is 2%. If you don't have llama-flu, the probability the test is negative is 95%. The most important question after all of this: if you test positive, what is the probability you have llama-flu?

Solution:

In the below, let L be the event of having llama-flu and $+$ be the event of testing positive. So, the probability of having llama-flu given we test positive is given by the following:

$$P(L|+) = \frac{P(+|L)P(L)}{P(+)} \text{ [bayes]}$$

$$= \frac{P(+|L)P(L)}{P(+|L)P(L) + P(+|L^C)P(L^C)} \text{ [ltp]}$$

$P(+|L) + P(-|L) = 1$, so $P(+|L) = 0.98$. (Given you have Llama-Flu, you either test positive or negative).

$P(+|L^C) + P(-|L^C) = 1$, so $P(+|L^C) = 0.05$. (Same for not having Llama-Flu)

$$= \frac{0.98 \cdot 0.001}{0.98 \cdot 0.001 + 0.05 \cdot 0.999} \approx 0.01924$$

Let's interpret this, given that we test positive, the probability we have the disease is only 0.01924 (~2%). Why is this true? Because the probability of having the disease was initially very low. Actually, the probability of having the disease given you test positive is 19-fold increase over the probability of having the disease! Meaning it's 19-times more that likely you have the disease given you test positive than just that you have the disease.

$$\frac{P(L|+)}{P(L)} = \frac{0.01924}{0.001} = 19.24$$

[**Tags**: Equally Likely Outcomes, Bayes Theorem, Law of Total Probability]

2.

Suppose we have three urns with the following number of red, white, and blue balls in them:

Urn	Red	White	Blue
A	6	5	2
B	4	3	6
C	5	6	7

Suppose we choose an urn by the following rules, after flipping a fair coin three times independently:

(a) If all flips are the same, pick from Urn A

(b) If there is exactly one head, pick from Urn B

(c) Else, pick from Urn C

After choosing an urn, we draw 5 balls without replacement, and let R be the event that exactly three of them are red. Let A, B, C be the events we chose urn A,B,C respectively. What is the probability we chose urn C, given that we drew exactly three of the five balls being red? We'll solve this in three steps.

(a) First, find $\Pr(A), \Pr(B), \Pr(C)$.

(b) Now find $\Pr(R)$, and do not simplify.

(c) Finally, compute $\Pr(C \mid R)$, and do not simplify.

Solution:

(a) First to find $\Pr(A)$, $\Pr(B)$ and $\Pr(C)$ we can consider that the $|\Omega|$ is 8 since there are 3 coin flips, each which can be either heads or tails. So, we have $2 \times 2 \times 2 = 2^3 = 8$. Then $|A| = 2$ since there are two ways to get all three coins the same, either all heads or all tails. Then $|B| = \binom{3}{1} = 3$ since there are three flips and we are choosing one to be heads, that is we can have (HTT, THT, or TTH). Because these are equally likely outcomes, we know that $P(A) = \frac{2}{8}$ and $P(A) = \frac{3}{8}$. Finally, $P(A) + P(B) + P(C) = 1$ because A, B and C partition the sample space. This means that $P(C) = 1 - \frac{2}{8} - \frac{3}{8} = \frac{3}{8}$.

(b) First note that that by the law of total probability:

$$P(R) = P(R \mid A)P(A) + P(R \mid B)P(B) + P(R \mid C)P(C)$$

We solved for each of $P(A), P(B)$ and $P(C)$ in part (a).

To calculate $P(R|A)$ we can also use equally likely outcomes and divide the number of ways to choose 3 red balls and 2 other balls from Urn A by the number of ways to choose 5 balls from Urn A. The number of ways to choose 3 red balls and 2 other balls is given by $\binom{6}{3}\binom{7}{2}$ because there are six red balls to choose the 3 from and 7 other balls to choose the 2 from. The number of ways to choose 5 balls from the urn is $\binom{13}{5}$. So, $P(R \mid A) = \frac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}}$.

We can use the same logic to calculate $P(R \mid B) = \frac{\binom{4}{3}\binom{9}{2}}{\binom{13}{5}}$ and $P(R \mid C) = \frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}}$.

This gives us the following as our final answer:

$$P(R) = P(R \mid A)P(A) + P(R \mid B)P(B) + P(R \mid C)P(C)$$

$$= \frac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}} \cdot \frac{2}{8} + \frac{\binom{4}{3}\binom{9}{2}}{\binom{13}{5}} \cdot \frac{3}{8} + \frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}$$

(c) To solve for $P(C \mid R)$ we can use Bayes Theorem:

$$P(C \mid R) = \frac{P(R \mid C)P(C)}{P(R)}$$

We have solved for all these quantities in the above, so we have:

$$P(C \mid R) = \frac{P(R \mid C)P(C)}{P(R)} = \frac{\frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}}{\frac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}} \cdot \frac{2}{8} + \frac{\binom{4}{3}\binom{9}{2}}{\binom{13}{5}} \cdot \frac{3}{8} + \frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}}$$

For more examples and solutions (S01 and S02), see https://courses.cs.washington.edu/courses/cse312/18sp/.