CSE 312: Foundations of Computing II
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Lecture Topics: 1.3 No More Counting Please

[**Tags**: PSet1 Q7, Combinatorial Proofs]

1. **Give combinatorial proofs of the following identities:**
   a. Prove that $\binom{n}{2} = \sum_{k=1}^{n-1} k$.
   b. Prove that $2^n - 1 = \sum_{i=0}^{n-1} 2^i$. (**Hint**: Imagine a tournament bracket)

Solution: Watch lecture ☺

[**Tags**: Complementary Counting, Inclusion-Exclusion, Multinomial Coefficients]

2. Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other (there are two E's, two N's, and two I's). For example, "INGREEDINT" is invalid because the two E's are adjacent. Repeat the question for the letters "AAAAABBB".

Solution: The overall set of ways to arrange INGREDIENT without restrictions is $|U| = \frac{10!}{2!2!2!}$, since we have three duplicates.

Let S be the set of those ways where no two identical letters are adjacent (this is what we want). Then S^C is the set of those ways where at least two identical letters are adjacent (meaning at least one of EE exists, NN exists, II exists). We shall find S by complementary counting: $|S| = |U| - |S^C|$, and we already have $|U|$ above.

If we let
T_E: the arrangements where the two E's are adjacent
T_N: the arrangements where the two N's are adjacent
T_I: the arrangements where the two I's are adjacent

Then, $S^C = T_E \cup T_N \cup T_I$. By inclusion-exclusion, (singles – doubles + triples – quads …)

$$|S^C| = |T_E| + |T_N| + |T_I| - |T_E \cap T_N| - |T_E \cap T_I| - |T_N \cap T_I| + |T_E \cap T_N \cap T_I|$$

Note that $|T_E| = |T_N| = |T_I|$ since there are two of each (so it's the same logic). Let's compute $|T_E|$: we must have both E's together, so let's treat them as one entity. Then, there are 9 entities total, so we get

$$\frac{9!}{2!\,2!}$$

Same for the doubles, they are all the same quantity. Let's compute $|T_E \cap T_N|$: we must have both E's and both N's together, so let's treat them as one entity. Then, there are 8 entities total, so we get

$$\frac{8!}{2!}$$

Finally, $|T_E \cap T_N \cap T_I|$ has 7 (distinct) entities, with $7!$ arrangements. Hence, our answer is

$$|S| = |U| - |S^C| = \frac{10!}{2!\,2!\,2!} - \left(3 \cdot \frac{9!}{2!\,2!} - 3 \cdot \frac{8!}{2!} + 7!\right)$$

For more examples and solutions, see https://courses.cs.washington.edu/courses/cse312/18sp/.