CSE 312: Foundations of Computing II
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Lecture Topics: 5.3 Conditional Distributions

[**Tags:** Conditional Expectation, Law of Total Expectation]

1. Suppose $X \sim Unif(0,1)$ (continuous uniform). We repeatedly draw iid $Y_1, Y_2, Y_3, \ldots \sim Unif(0,1)$ (continous uniform) until the first random time T that $Y_T < X$. What is $E[T]$?

We could try to find the PMF of T using the LTP, after determining $\Omega_T = \{1,2,3,\ldots\}$, but that's no fun. The PDF of X is actually:

$$f_X(x) = \begin{cases} \frac{1}{1-0} = 1, & 0 \leq x \leq 1 \\ 0, & 0 > x, 1 < x \end{cases}$$

So, we actually only need to integrate from 0 to 1 and the PDF of X will be 1.

Given X = x, then the probability that T=t will be $x(1-x)^{t-1}$ because the probability of getting something that is less than x is x which will happen once (at the end) and the probability of getting something greater than x is $1-x$ which will happen t-1 times. So, we have:

$$P(T=t) = \int_0^1 P(T=t|X=x) f_X(x) dx = \int_0^1 x(1-x)^{t-1} \cdot 1 dx = \left[-\frac{(1-x)^t(tx+1)}{t(t+1)} x^t\right]_0^1$$
$$= \frac{0}{t(t+1)} - \left(-\frac{1}{(t)t+1}\right) = \frac{1}{t(t+1)}$$

Which gives us:

$$E[T] = \sum_{t=1}^{\infty} t p_T(t) = \sum_{t=1}^{\infty} t \frac{1}{t(t+1)} = \sum_{t=1}^{\infty} \frac{1}{t+1} = \infty$$

Alternatively and preferably, by LTE, we have that $(T|X=x) \sim Geo(x)$, so $E[T|X=x] = \frac{1}{x}$.

$$E[T] = \int_0^1 E[T|X=x] f_X(x) dx = \int_0^1 \frac{1}{x} dx = \infty$$

[**Tags:** **<u>PSet4 Q1a</u>**, Conditional Distributions]

2. Suppose $X \sim Bin(n,p)$ and $Y \sim Bin(m,p)$ are independent, and let $Z = X + Y$. What is the conditional PMF $P(X=k|Z=z)$? Actually, $X|Z=z$ is a parametrized distribution we know. What is its name and what are its parameter(s)? (Hint: You know the distribution of Z, and can look up its PMF!)

<u>Solution</u>: Watch lecture ☺ .

[**Tags:** Law of Total Probability/Expectation, Conditional Expectation]

3. Suppose the number of radioactive particles emitted in an hour are $X \sim Poi(\lambda)$. You have a device which records each particle emission with probability p (ideally close to 1), independent of other particles. Let Y be the number of particles actually observed by the device. What is $E[Y]$?

Solution:

Note that $(Y|X=x) \sim Bin(x,p)$, so $E[Y|X=x] = xp$.

$$\begin{aligned} E[Y] &= \sum_{x=0}^{\infty} E[Y|X=x] p_X(x) \text{ [LTE]} \\ &= \sum_{x=0}^{\infty} xp \cdot p_X(x) \\ &= p \sum_{x=0}^{\infty} x p_X(x) \\ &= pE[X] \\ &= p\lambda \end{aligned}$$

Sometimes, people write LTE as $E[Y] = E_X\left[E_{Y|X}[Y|X]\right]$, which is supposed to equal

$$\sum_{x \in \Omega_X} E_{Y|X}[Y|X=x] p_X(x)$$

And so, $Y|X \sim Bin(X,p)$ (just another way to write $(Y|X=x) \sim Bin(x,p)$), and so $E_{Y|X}[Y|X] = Xp$,

$$E[Y] = E_X\left[E_{Y|X}[Y|X]\right], = E_X[Xp] = pE_X[X] = p\lambda$$

I don't like this notation though!