CSE 312: Foundations of Computing II
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Lecture Topics: 4.2 Zoo of Continuous Random Variables

[**Tags:** PDFs, CDFs, Exponential, Uniform]

1. You are waiting for a bus to take you home from CSE. You can either take the E-line, U-line, or C-line. The distribution of the waiting time in minutes for each is the following:
   - E-Line: $E \sim Exp(\lambda = 0.1)$
   - U-Line: $U \sim Unif(0, 20)$ (continuous)
   - C-line: Has range $(1, \infty)$ and density function $f_C(x) = 1/x^2$.

   Assume the three bus arrival times are independent. You take the first bus that arrives.

   a. Find the CDF's of $E, U,$ and C, $F_E(t), F_U(t)$, and $F_C(t)$. **Hint**: The first two can be looked up in a table.
   b. What is the probability you wait more than 5 minutes for a bus?
   c. What is the probability you wait more than 30 minutes for a bus?
   d. (Challenge) What is the expected amount of time you will wait for a bus? **Hint**: Compute the CDF first which has four parts: $(-\infty, 0]$, $(0,1]$, $(1,20]$, and $(20, \infty)$.

Solution:

a. The CDF of E for $t > 0$ is $F_E(t) = P(X \le t) = 1 - e^{-0.1t}$.

   The CDF of U for $t > 0$ is $F_U(t) = \frac{t}{20}$.

   The CDF of C for $t > 1$ is $F_C(t) = \int_1^t f_C(x)dx = 1 - \frac{1}{t}$.

b. Let $B = \min\{E, U, C\}$ be the time until the first bus.

$$P(B > 5) = P(E > 5, U > 5, C > 5) = P(E > 5)P(U > 5)P(C > 5)$$

$$= (1 - F_E(5))(1 - F_U(5))(1 - F_C(5)) = e^{-0.5} \cdot \frac{15}{20} \cdot \frac{1}{5} = \frac{3}{20} e^{-0.5}$$

c. This probability is 0, since the range of U is $[0,20]$, and is guaranteed to come within 20 minutes.

d. This gets quite messy, but the CDF is:

$$F_B(t) = P(B \le t) = 1 - P(B > t) = 1 - P(E > t)P(U > t)P(C > t)$$

   So, since for any t less than 0, each of E, U, and C will all be greater than t, for t between 0 and 1, the probability C is greater than t is 1, and for t greater than 20, at least U will have come, we have:

$$F_B(t) = \begin{cases} 1 & t \le 0 \\ (e^{-t})(1 - \frac{t}{20}) & 0 < t \le 1 \\ (e^{-t})(1 - \frac{t}{20})(\frac{1}{t}) & 1 < t \le 20 \\ 0 & t > 20 \end{cases}$$

   Which implies that, by taking the derivative for the CDF, we have the following for the PDF:

$$f_B(t) = \begin{cases} 0 & t \leq 0 \\ \frac{e^{-t}}{20}(-21+t) & 0 < t \leq 1 \\ (e^{-t})\frac{-20-20t+t^2}{20t^2} & 1 < t \leq 20 \\ 0 & t > 20 \end{cases}$$

So, for the expected value we have:

$$\mathbb{E}[B] = \int_0^1 t[\frac{e^{-t}}{20}(-21+t)]dt + \int_1^{20} t[(e^{-t})\frac{-20-20t+t^2}{20t^2}]dt$$

[**Tags:** PSet3 Q5, Exponential, Memorylessness, Gamma]

2. You have n batteries, each with a lifetime which is (independently) distributed as $Exp(\lambda)$. You have a choice of a weak flashlight, which requires one battery to operate, and a strong flashlight, which requires two batteries to operate. Assume that when a battery dies, you are lightning-quick and replace it with a new battery instantly.
    a. If you choose to use the weak flashlight, what is the expected amount of time you can operate it for? (Hint: Cite the appropriate distribution, and your solution will be one-line.)
    b. Recall the memoryless property in lecture 4.2. Suppose $W \sim Exp(\beta)$. Show that you understand what it means by computing $P(W > 17 | W > 10)$ explicitly using this property (do NOT reprove memorylessness).
    c. For the strong flashlight, we need to compute the distribution of time that until the first of the two batteries dies. If $X, Y \sim Exp(\lambda)$, show that the distribution of $Z = \min\{X, Y\}$ is $Exp(2\lambda)$. (Hint: Start by computing $P(Z > z)$, then use this to compute either the CDF or PDF).
    d. Left for you!

Solution: Watch lecture ☺