Chapter 1. Combinatorial Theory

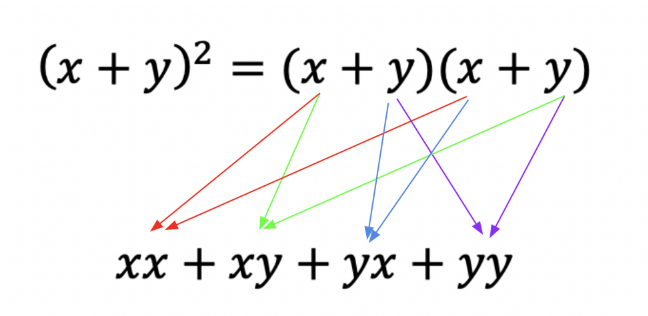
1.3: No More Counting Please

Slides (Google Drive) Alex Tsun Video (YouTube)

In this section, we don't really have a nice successive ordering where one topic leads to the next as we did earlier. This section serves as a place to put all the final miscellaneous but useful concepts in counting.

1.3.1 Binomial Theorem

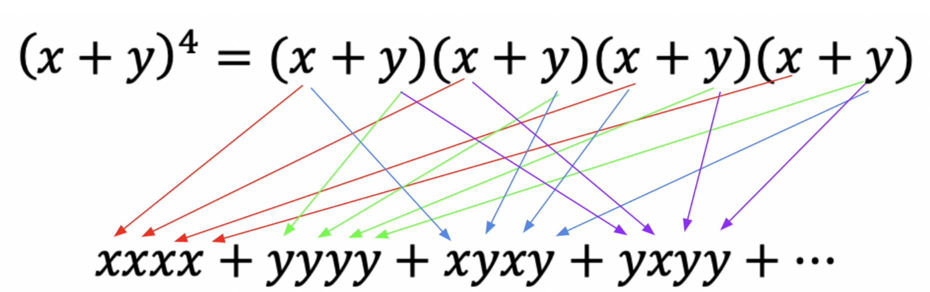
We talked last time about binomial coefficients of the form $\binom{n}{k}$. Today, we'll see how they are used to prove the binomial theorem, which we'll use more later on. For now, we'll see how they can be used to expand (possibly large) exponents below. You may have learned this technique of FOIL (first, outer, inner, last) for expanding $(x+y)^2$.



We then combine like-terms (xy and yx).

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y) \\ &= xx + xy + yx + yy && \text{[FOIL]} \\ &= x^2 + 2xy + y^2\end{aligned}$$

But, let's say that we wanted to do this for a binomial raised to some higher power, say $(x+y)^4$. There would be a lot more terms, but we could use a similar approach.



$$
\begin{aligned}
(x+y)^4 &= (x+y)(x+y)(x+y)(x+y) \\
&= xxxx + yyyy + xyxy + yxyy + \dots
\end{aligned}
$$

But what are the terms exactly that are included in this expression? And how could we combine the like-terms though?

Notice that each term will be a mixture of x's and y's. In fact, each term will be in the form $x^k y^{n-k}$ (in this case $n = 4$). This is because there will be exactly n x's or y's in each term, so if there are k x's, then there must be $n - k$ y's. That is, we will have terms of the form $x^4, x^3y, x^2y^2, xy^3, y^4$, with most appearing more than once.

For a specific k though, how many times does $x^k y^{n-k}$ appear? For example, in the above case, take $k = 1$, then note that $xyyy = yxyy = yyxy = yyyx = xy^3$, so xy^3 will appear with the coefficient of 4 in the final simplified form (just like for $(x + y)^2$ the term xy appears with a coefficient 2). Does this look familiar? It should remind you yet again of rearranging words with duplicate letters!

Now, we can generalize this, as the number of terms will simplify to $x^k y^{n-k}$ will be equivalent to the number of ways to choose exactly k of the binomials to give us x (and let the remaining $n-k$ give us y). Alternatively, we need to arrange k x's and $n - k$ y's. To think of this in the above example with $k = 1$ and $n = 4$, we were consider which of the four binomials would give us the single x, the first, second, third, or fourth, for a total of $\binom{4}{1} = 4$.

Let's consider $k = 2$ in the above example. We want to know how many terms are equivalent to x^2y^2. Well, we then have $xxyy = yxxy = yyxx = xyxy = yxyx = xyyx = x^2y^2$, so there are six ways and the coefficient on the simplified term x^2y^2 will be $\binom{4}{2} = 6$.

Notice that we are essentially choosing which of the binomials gives us an x such that k of the n binomials do. That is, the coefficient for $x^k y^{n-k}$ where k ranges from 0 to n is simply $\binom{n}{k}$. This is why it was also called a binomial coefficient.

That leads us to the binomial theorem:

Theorem 1.3.1: Binomial Theorem

Let $x, y \in \mathbb{R}$ be real numbers and $n \in \mathbb{N}$ a positive integer. Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

This essentially states that in the expansion of the left side, the coefficient of the term with x raised to the power of k and y raised to the power of $n - k$ will be $\binom{n}{k}$, and we know this because we are considering the number of ways to choose k of the n binomials in the expression to give us x.

This can also be proved by induction, but this is left as an exercise for the reader.

Example(s)

Calculate the coefficient of $a^{45}b^{14}$ in the expansion $(4a^3 - 5b^2)^{22}$.

Solution Let $x = 4a^3$ and $y = -5b^2$. Then, we are looking for the coefficient of $x^{15}y^7$ (because x^{15} gives us

a^{45} and y^7 gives us b^{14}), which is $\binom{22}{15}$. So we have the term

$$\binom{22}{15} x^{15} y^7 = \binom{22}{15} (4a^3)^{15} (-5b^2)^7 = \left(-\binom{22}{15} 4^{15} 5^7\right) a^{45} b^{14}$$

and our answer is $-\binom{22}{15} 4^{15} 5^7$. □

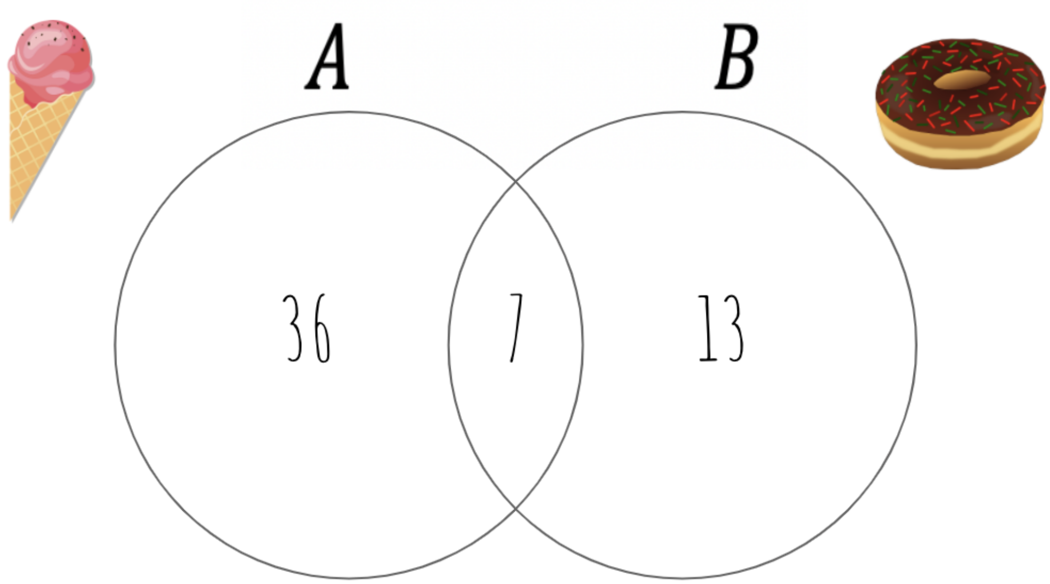
1.3.2 Inclusion-Exclusion

Say we did an anonymous survey where we asked whether students in CSE312 like ice cream, and found that 43 people liked ice cream. Then we did another anonymous survey where we asked whether students in CSE312 liked donuts, and found that 20 people liked donuts. With this information can we determine how many people like ice cream or donuts (or both)?

Let A be the set of people who like ice cream, and B the set of people who like donuts. The sum rule from 1.1 said that, if A, B were mutually exclusive (it wasn't possible to like both donuts and ice cream: $A \cap B = \emptyset$), then we could just add them up: $|A \cup B| = |A| + |B| = 43 + 20 = 63$. But this is not the case, since it is possible that to like both. We can't quite figure this out yet without knowing how many people overlapped: the size of $A \cap B$.

So, we did another anonymous survey in which we asked whether students in CSE312 like both ice cream and donuts, and found that only 7 people like both. Now, do we have enough information to determine how many students like ice cream or donuts?

Yes! Knowing that 43 people like ice cream and 7 people like both ice cream and donuts, we can conclude that 36 people like ice cream but don't like donuts. Similarly, knowing that 20 people like donuts and 7 people like both ice cream and donuts, we can conclude that 13 people like donuts but don't like ice cream. This leaves us with the following picture, where A is the students who like ice cream. B is the students who like donuts (this implies $|A \cap B| = 7$ is the number of students who like both):



So we have the following:

$$|A| = 43$$
$$|B| = 20$$
$$|A \cap B| = 7$$

Now, to go back to the question of how many students like either ice cream or donuts, we can just add up the 36 people that just like ice cream, the 7 people that like both ice cream and donuts, and the 13 people that just like donuts, and get $36 + 7 + 13 = 56$. Alternatively, we could consider this as adding up the 43 people who like ice cream (including both the 36 those who just like ice cream and the 7 who like both) and the 20 people who like donuts (including the 13 who just like donuts and the 7 who like both) and then subtracting the 7 who like both since they were counted twice. That is $43 + 20 - 7 = 56$. That leaves us with:

$$|A \cup B| = 36 + 7 + 13 = 56 = 43 - 20 - 7 = |A| + |B| - |A \cap B|$$

Recall that $|A \cup B|$ is the students who like donuts or ice cream (the union of the two sets).

Theorem 1.3.2: Inclusion-Exclusion

Let A, B be sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Further, in general, if $A_1, A_2, \ldots, A_n$ are sets, then:

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| &= \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \cdots \\ &= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) \\ &\quad + (A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) + \cdots \end{aligned}$$

where singles are the sizes of all the single sets ($\binom{n}{1}$ terms), doubles are the sizes of all the intersections of two sets ($\binom{n}{2}$ terms), triples are the size of all the intersections of three sets ($\binom{n}{3}$ terms), quads are all the intersection of four sets, and so forth.

Example(s)

How many numbers in the set $[360] = \{1, 2, \ldots, 360\}$ are divisible by:

1. 4, 6, and 9.
2. 4, 6 or 9.
3. neither 4, 6, nor 9.

Solution

1. This is just the multiplies of $\text{lcm}(4, 6, 9) = 36$, which there are $\dfrac{360}{36} = 10$ of.
2. Let D_i be the number of numbers in $[360]$ which are divisible by i, for $i = 4, 6, 9$. Hence, the number of numbers which are divisible by 4, 6, or 9 is $|D_4 \cup D_6 \cup D_9|$. We can apply inclusion-exclusion (singles

minus doubles plus triples):

$$\begin{aligned} |D_4 \cup D_6 \cup D_9| &= |D_4| + |D_6| + |D_9| - |D_4 \cap D_6| - |D_4 \cap D_9| - |D_6 \cap D_9| + |D_4 \cap D_6 \cap D_9| \\ &= \frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{12} - \frac{360}{36} - \frac{360}{18} + \frac{360}{36} \end{aligned}$$

Notice the denominators for the paired terms are again, dividing by the least common multiple.

3. Complementary counting - this is just 360 minus the answer from the previous part!

□

Many times it may be possible to avoid this ugly mess using complementary counting, but sometimes it isn't.

1.3.3 Pigeonhole Principle

The Pigeonhole Principle is a tool that allows us to make guarantees when we tackle problems like: if we want to assign 20 third-grade students to 3 (equivalent) classes, how can we minimize the largest class size? It turns out we can't do any better than having 7 people in the largest class. The reason is because of the pigeonhole principle!

We'll start with a smaller but similar problem. If 11 children have to share 3 beds, how can we minimize the number of children on the most crowded bed? The idea might be just to spread them "uniformly". Maybe number the beds A,B,C, and assign the first child to A, the second to B, the third to C, the fourth to A, and so on. This turns out to be optimal as it spreads the kids out as evenly as possible. The pigeonhole principle tells us the best worst-case scenario: that at least one bed must have at least 4 children.

You might first distribute the children evenly amongst the beds, say put 3 children in each bed to start. That leaves us with 3 times 3 equals 9 children accounted for, and 2 children remaining with a bed. Well, they must be put to bed, so we can put each of them in a separate bed and we finish with the first bed having 4, the second bed having 4, and the third bed having 3. No matter how we move the children around, we can't have an an arrangement where at least one bed will contain at least 4 children.

We could also have found this by dividing 11 by 3 and rounding up to account for the remainder (which must go somewhere). Before formally defining the pigeonhole principle, we need to define the floor and ceiling functions.

Definition 1.3.1: Floor and Ceiling Functions

The **floor** function $\lfloor x \rfloor$ returns the largest integer $\leq x$ (i.e. rounds down).
The **ceiling** function $\lceil x \rceil$ returns the smallest integer $\geq x$ (i.e. rounds up). Note the difference is just whether the bracket is on top (ceiling) or bottom (floor).

Example(s)

Solve the following: $\lfloor 2.5 \rfloor$, $\lfloor 16.999999 \rfloor$, $\lfloor 5 \rfloor$, $\lceil 2.5 \rceil$, $\lceil 9.000301 \rceil$, $\lceil 5 \rceil$.

Solution

$$\lfloor 2.5 \rfloor = 2 \qquad \lfloor 16.999999 \rfloor = 16 \qquad \lfloor 5 \rfloor = 5$$
$$\lceil 2.5 \rceil = 3 \qquad \lceil 9.000301 \rceil = 10 \qquad \lceil 5 \rceil = 5$$

□

Theorem 1.3.3: Pigeonhole Principle

If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 pigeons.
More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

This fact or rule may seem trivial to you, but the hard part of pigeonhole problems is knowing how to apply it. See the examples below!

Example(s)

Show that there exists a number made up of only 1's (e.g., 1111 or 11) which is divisible by 333.

Solution Consider the sequence of 334 numbers $x_1, x_2, x_3, \dots, x_{334}$ where x_i is the number made of exactly i 1's (e.g., $x_2 = 11$, $x_5 = 11,111$, etc.). We'll use the notation $x_i = 1^i$ to mean i 1's concatenated together.

The number of possible remainders when dividing by 333 is 333: $\{0, 1, 2, \dots, 332\}$, so by the pigeonhole principle, since $334 > 333$, two numbers x_i and x_j have the same remainder (suppose $i < j$ without loss of generality) when divided by 333. The number $x_j - x_i$ is of the form $1^{j-i}0^i$; that is $j - i$ 1's followed by i 0's (e.g., $x_5 - x_2 = 11111 - 11 = 11100 = 1^3 0^2$). This number must be divisible by 333 because $x_i \equiv x_j \pmod{333} \Rightarrow (x_j - x_i) \equiv 0 \pmod{333}$.

Now, keep deleting zeros (by dividing by 10) until there aren't any more left - this doesn't affect whether or not 333 goes in since neither 2 nor 5 divides 333. Now we're left with a number divisible by 333 made up of all ones (1^{j-i} to be exact)!

Note that 333 was not special - we could have used any number that wasn't divisible by 2 nor 5. □

Example(s)

Show that in a group of n people (who may be friends with any number of other people), two must have the same number of friends.

Solution We have two cases.

1. Case 1: Everyone has at least one friend. Then, everyone has a number of friends between $1, 2, \dots, n-1$. By the pigeonhole principle, since there are n people and $n - 1$ possibilities, at least two people have the same number of friends.

2. Case 2: At least one person has no friends. Let's take one such person and call them A. Then, the other $n-1$ people can have number of friends from $0, \dots, n-2$ since they can't be friends with A. We have two more cases within this case unfortunately.

   (a) Case 2a: If one of these $n - 1$ people has no friends, we are done since A and this person both have 0 friends.

   (b) Case 2b: Otherwise, these people all have at least one friend, from $1, \dots, n - 2$, and since there are $n - 1$ people and $n - 2$ possibilities, at least two people have the same number of friends.

In all cases we are guaranteed that two people have the same number of friends.

□

1.3.4 Combinatorial Proofs

You may have taken a discrete mathematics/formal logic class before this if you are a computer science major. If that's the case, you would have learned how to write proofs (e.g., induction, contradiction). Now that we know how to count, we can actually prove some algebraic identities using counting instead!

Suppose we wanted to show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ was true for any positive integer $n \in \mathbb{N}$ and $0 \leq k \leq n$.

We could start with an algebraic approach and try something like:

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)}{(k-1)!(n-k)!} + \frac{(n-1)}{k!(n-1-k)!} && \text{[def of binomial coef]} \\ &\cdots && \text{[lots of algebra]} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

However, those $\cdots$ may be tedious and take a lot of algebra we don't want to do.

So, let's consider another approach. A combinatorial proof is one where you prove two quantities are equal by imagining a situation/something to count. Then, you argue that the left side and right side are two equivalent ways to count the same thing, and hence must be equal. We've seen earlier often how there are multiple approaches to counting!

In this case, let's consider the set of numbers $[n] = \{1, 2, \ldots n\}$. We will argue that the LHS and RHS both count the number of subsets of size k.

1. LHS: $\binom{n}{k}$ is literally the number of subsets of size k, since we just want to choose any k items out of n (order doesn't matter).
2. RHS: We take a slightly more convoluted approach, splitting on cases depending on whether or not the number 1 was included in the subset.

   Case 1: Our subset of size k includes the number 1. Then we need to choose $k-1$ of the remaining $n-1$ numbers (n numbers excluding 1 is $n-1$ numbers) to make a subset of size k which includes 1.

   Case 2: Our subset of size k does not include the number 1. Then we need to choose k numbers from the remaining $n-1$ numbers. There are $\binom{n-1}{k}$ ways to do this. So, in total we have $\binom{n-1}{k-1} + \binom{n-1}{k}$ possible subsets of size k.

Since the left side and right side count the same thing, they must be equal! Note that we dreamed up this situation and you may wonder how we did - this just comes from practicing many types of counting problems. You'll get used to it!

Definition 1.3.2: Combinatorial Proofs

To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

Example(s)

Prove the following two identities combinatorially (NOT algebraically):

1. Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.
2. Prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Solution

1. We'll show that both sides count, from a group of n people, the number of committees of size m, and within that committee a subcommittee of size k.

   Left-hand side: We first choose m people to be on the committee from n total; there are $\binom{n}{m}$ ways to do so. Then, within those m, we choose k to be on a specialized subcommittee; there are $\binom{m}{k}$ ways to do so. By the product rule, the number of ways to assign these is $\binom{n}{m}\binom{m}{k}$.

   Right-hand side: We first choose which k to be on the subcommitee of size k; there are $\binom{n}{k}$ ways to do so. From the remaining $n - k$ people, we choose $m - k$ to be on the committee (but not the subcommittee). By the product rule, the number of ways to assign these is $\binom{n}{k}\binom{n-k}{m-k}$.

   Since the LHS and RHS both count the same thing, they must be equal.

2. We'll argue that both sides count the number of subsets of the set $[n] = \{1, 2, \ldots, n\}$.

   Left-hand side: Each element we can have in our subset or not. For the first element, we have 2 choices (in or out). For the second element, we also have 2 choices (in or out). And so on. So the number of subsets is 2^n.

   Right-hand side: The subset can be of any size ranging from 0 to n, so we have a sum. Now how many subsets are there of size exactly k? There are $\binom{n}{k}$ because we choose k out of n to have in our set (and order doesn't matter in sets)! Hence, the number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$

   Since the LHS and RHS both count the same thing, they must be equal.

   It's cool to note we can also prove this with the binomial theorem setting $x = 1$ and $y = 1$ - try this out! It takes just one line!

□

1.3.5 Exercises

1. These problems involve using the pigeonhole principle. How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:

(a) A single pair? (e.g., AA, 99, JJ)

(b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)

(c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)

(d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?

(e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)

(f) A straight flush (5 cards which are both a straight and a flush)?

Solution:

(a) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is 14.

(b) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is 17.

(c) The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is 27.

(d) The worst that could happen is to draw all the A - 4, 6 - 9, and J - K. After drawing these $11 \cdot 4 = 44$ cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is 45.

(e) The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is 17.

(f) Same as straight, 45.